Solutions of the continuity equation

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Solutions are obtained to the continuity equation for the transport of a quantity \( f(r, t) \) by a velocity \( v \) or the change of a distribution \( n(r, t) \) by a growth rate \( \dot{r} \). When \( v \) is a separable function of \( r \) and \( t \), explicit solutions are obtained from the formal solutions known in hydrodynamics theory which are extended to include contributions from source terms. When \( v \) is also a function of \( f \), a general method is given for solving the equation, together with an illustrative example referring to porosity transport in nuclear fuel elements. In the case when \( \dot{r} \) depends on moments of the distribution, a method of solution is suggested which involves an expansion in terms of the moments and which can be applied to void swelling in steels. Perturbation theory and the application of the results to more general problems are discussed.

1. Introduction

The continuity equation for the transport of a density \( \rho \) by a velocity \( \bm{v} \) is one of the most familiar equations of theoretical physics,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bm{v}) = 0.
\]

Originally it was known as the spatial continuity equation of D'Alembert & Euler (see Truesdale & Toupin 1960). In a different context the continuity equation for the rate of growth of particles in solution was derived as (Lialikov et al. 1935)

\[
\frac{\partial n}{\partial t} + \frac{\partial (n \dot{r})}{\partial r} = 0,
\]

where \( n(r, t) \) is the number density of particles.

Evidently, in one dimension with \( v = \dot{r} \), the two equations are identical in form. The aim of this work is to discuss solutions to these and similar equations with and without additional source terms.

In the context of hydrodynamics most of the solutions we write down are given in terms of the so-called Lagrangian variables although they were apparently originally discussed by Euler (Lamb 1932). In older books on fluid mechanics such as Lamb (1932) there is some discussion of the solutions, but in more recent books, for example Landau & Lifshitz (1959), they are hardly mentioned. The reason for this neglect is that the main problem in hydrodynamics is the determination of \( \bm{v} \) for which Lagrangian variables, which represent positions along a trajectory or path,
are not suitable. However, in many other problems in physics and chemistry $\nu$ or $\dot{\nu}$ may be regarded as a given function. For example, in many cases the motion of macroscopic objects such as pores in solids or bubbles in liquids is directional and not diffusional and is determined by external quantities such as the temperature gradient. Some special cases of the solutions given here have been discussed with reference to lenticular pore motion up a temperature gradient by Clement (1977). Although they are not new, the basic solutions given in §2a for one-dimensional motion and in §3 for three-dimensional motion have not been widely applied. Indeed in such well-known books as Margenau & Murphy (1956) on the mathematics of physics and chemistry there is no mention of them.

In §2b and §3 we obtain explicit solutions when there are source terms present. In the case of directional motion these terms can arise from the formation of macroscopic objects such as pores and in the growth case they can represent nucleation terms. It is therefore expected that they will have widespread applicability.

So far solutions have been referred to where $\nu$ and $\dot{\nu}$ are functions of $r$ and $t$ so that equations (1) and (2) are linear in $\rho$ and $n$, respectively. The more general nonlinear case in which, the velocity or growth rate is a function of $\rho$ or $n$ is discussed in §4. Where there is a direct functional dependence, the equations take the general form of the class of quasilinear equations discussed by Courant & Hilbert (1962). Their main method of solution is briefly given in §4a and it is shown how it can be applied to find the solution of a specific nonlinear pore transport equation (Clement & Finnis 1977) in §4b. For the growth of a distribution of macroscopic objects, the growth rate $\dot{\nu}$ generally turns out to be a function of time through moments of the distribution of $n$. As far as is known, the general solution of the resulting integro-differential equation has not received attention from mathematicians or, if it has the results are not readily accessible to physicists and chemists. The special case of the Ostwald ripening of small particles has received much attention (see, for example, Wagner 1961; Dunning 1973a, b) though the mathematics used is not always very rigorous. Another important case of physical interest occurs in the rate theory of swelling from void growth in irradiated metals (Brailsford & Bullough 1972) where $\dot{\nu}$ is a function of the mean size of void. In §4.3 it is shown how the solutions derived hitherto can be used to tackle this problem.

Finally in §5 it is shown how perturbation theory can be applied to the equation when $\nu$ is the sum of two terms, one of which is small. The conditions under which perturbation theory can be applied to a small diffusive term are also briefly examined. Some discussion is given as to the use of the formal solutions in continuity equations which contain collision or coagulation terms. In the conclusions the practical utility of the solutions and methods of solution contained in this paper are emphasized.
2. Solutions in One Space Dimension

With the inclusion of a source term, \( S(r, t) \), we rewrite the conservation equation as

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial r} [v(r, t)f] = S(r, t).
\]  

(3)

The same equation and the resulting solutions apply in the cylindrically symmetric and spherically symmetric cases in two and three dimensions, if we replace \( f \) and \( S \) by \( rf \) and \( rS \) and by \( r^2f \) and \( r^2S \), respectively. We first consider the case of no source term and then give solutions involving source terms which include those from a boundary in space. By a change of sign, \( S \) can also represent a sink but then we must not generally allow \( f \) to become negative if it is to represent a physical distribution.

The velocity \( v \), which may also be taken to represent \( i \), is assumed not to have mathematically singular properties which would enable, for example, the bifurcation of trajectories through a point.

(a) The general solution with no source term

We have already mentioned trajectories in connection with \( v(r, t) \). These are paths in \( r \) and \( t \) which would be followed by a particle moving with a velocity \( v \). It is intuitively obvious that a complete knowledge of these trajectories must enable \( f \) to be determined at any future time starting with an initial distribution \( f_0(r, t = 0) \). An initial point \( r_0 \) will move to a point \( r \) at time \( t \), and with \( r_0 \) fixed, the equation for \( r \) is of course

\[
(\frac{\partial r}{\partial t})_{r_0} = v(r, t).
\]  

(4)

The solution of this equation is \( r(r_0, t) \) but conversely we can regard the solution as giving the function \( r_0(r, t) \). From the properties of partial derivatives we have

\[
\left( \frac{\partial r}{\partial t} \right)_{r_0} = -\left( \frac{\partial r_0}{\partial t} \right)_r \left( \frac{\partial r_0}{\partial r} \right)_t.
\]  

(5)

Thus the partial differential equation which gives \( r_0 \) directly is

\[
\frac{\partial r_0}{\partial t} = -v(r, t) \frac{\partial r_0}{\partial r}.
\]  

(6)

In terms of the solution for \( r_0 \), the solution of (3) for any \( f \) is

\[
f(r, t) = f_0(r_0(r, t)) \frac{\partial r_0(r, t)}{\partial r}
= -\frac{f_0(r_0(r, t))}{v(r, t)} \frac{\partial r_0}{\partial t}(r, t).
\]  

(7)

It may be verified by direct substitution that this solution satisfies equation (3). Thus the problem of solving (3) is reduced to that of finding the trajectories or solving one of the equations (4) or (6). This may have to be done numerically but if a number of solutions for different values of \( f_0 \) are required, the solution (7) will give them all in terms of one solution for \( r_0 \).
We shall use as our standard example the case when \( v \) is independent of \( t \); then the solution of (4) or (6) has the implicit form,

\[
t = \int_{r_0}^{r} \frac{dr'}{v(r')}. \tag{8}
\]

Here there is a symmetry in the trajectory between \( r \) and \( r_0 \) so that

\[
(\partial r_0/\partial t)_r = -v(r_0). \tag{9}
\]

The solution (7) is then

\[
f(r, t) = \left[ v(r_0)/v(r) \right] f_0(r_0(r, t)). \tag{10}
\]

In some situations of physical interest where an activation energy and a space-varying temperature specify \( v \) (Clement 1977) it is a good approximation to represent \( v \) over a range of \( r \) by an exponential,

\[
v(r) = v_0 \exp(-\lambda r). \tag{11}
\]

In this case we have explicitly

\[
r_0 = r \ln \left[ 1 - \frac{\lambda t}{v_0} \right], \tag{12}
\]

so that the solution for \( f \) may be obtained analytically.

An alternative way of writing the solution (10) is in the functional form (Clement 1977)

\[
f(r, t) = \frac{1}{v(r)} u \left( t - \int_{r_0}^{r} \frac{dr'}{v(r')} \right), \tag{13}
\]

where

\[
u \left( t - \int_{r_0}^{r} \frac{dr'}{v(r')} \right) = v(r)f_0(r). \tag{14}
\]

The most general, but not uncommon, case where the solution can easily be found in terms of integrals, is when \( v \) is separable,

\[
v(r, t) = g(t)v(r). \tag{15}
\]

Then equations (4) and (6) are separable and the above solutions (8) and (10) apply in terms of a new time variable defined by

\[
T = \int_{t_0}^{t} g(t) \, dt. \tag{16}
\]

(b) Solutions with source terms

It is evident that the general solution of equation (3) is the solution (7) which we have already obtained plus a particular solution involving the source term only, which we call \( h(r, t) \). First we consider a trajectory wholly within the spatial region where the equation applies, such that the trajectory passes through the points \((r_0, 0), (r_1, t_1)\), and \((r, t)\). Then \( r_1(r_0, t_1) \) satisfies the equation

\[
(\partial r_1/\partial t_1)_{r_0} = v(r_1, t_1), \tag{17}
\]

whose solution, since \( r_0 \) is a function of \((r, t)\), is \( r_1(r_0(r, t), t_1) \).
Solutions of the continuity equation

Now the function at \((r, t)\) receives contributions from all points along the path from \(r_0\) at \(t = 0\) according to equation (7). Thus we have

\[
h(r, t) = \int_0^t S(r_0(r, t), t_1) \frac{\partial r_1(r_0(r, t), t_1)}{\partial r} \, dt_1
\]

\[
= \frac{\partial r_0}{\partial r} (r, t) \int_0^t S(r_0(r, t), t_1) \frac{\partial r_1(r_0(r, t), t_1)}{\partial r_0} \, dt_1.
\]  

(18)

It may be verified directly that this solution satisfies equation (3) with the use of the functional identity,

\[
r_1(r_0(r, t), t) = r.
\]  

(19)

Alternatively we can integrate along the path in terms of the spatial coordinate \(r_1\) by substituting the equation

\[
\frac{\partial r_1}{\partial r_0} = - \frac{v(r_1, t_1)}{\partial r_0},
\]  

(20)

and changing the variable to \(r_1\):

\[
h(r, t) = \int_{r_0(r, t)}^{r} S(r_1, t_1(r_0(r, t), r_1)) \frac{\partial r_1}{\partial r_0} \, dr_1,
\]  

(21)

where now

\[
t_1(r_0(r, t), r) = t.
\]  

(22)

Again it is possible to prove directly with the use of equation (6) that the solution (21) satisfies equation (3). When \(v\) is independent of time, (21) reduces to

\[
h(r, t) = \frac{1}{v(r)} \int_{r_0(r, t)}^{r} S(r_1, \int_{r_0(r, t)}^{r_1} \frac{dr'}{v(r')} \, dr') \, dr_1.
\]  

(23)

So far we have considered sources which can act as boundaries in time. Now let us consider a boundary in space, sat at \(r = 0\), so that a trajectory passes through the points \((0, t_1), (r', t'),\) and \((r, t)\). Then from (18) the complete solution in the region affected by the boundary is

\[
h(r, t) = \int_{t_1}^{t} S(r'(t_1(r, t), t'), t') \frac{\partial r'}{\partial r} (t_1(r, t), t') \, dt'
\]

\[
= \frac{\partial t_1}{\partial r} \int_{t_1}^{t} S(r', t') \frac{\partial r'}{\partial t_1} \, dt',
\]  

(24)

where

\[
r'(t_1(r, t), t) = r.
\]  

(25)

Again this result may be transformed to an integral in space to give

\[
h(r, t) = - \frac{\partial t_1}{\partial r} \int_0^r S(r', t'(t_1(r, t))) \frac{\partial t_1}{\partial r} \, dr'.
\]  

(26)

As an example take a \(\delta\)-function source,

\[
S(r', t') = S(t') \delta(r').
\]  

(27)
The following results apply:

\[ t'(0, t_1(r, t)) = t_1, \frac{\partial t'}{\partial t_1} = 1, \]

so that (26) reduces to

\[ h(r, t) = -\left[ \frac{\partial t_1(r, t)}{\partial r} \right] S(t_1(r, t)). \]

If \( v \) is independent of time (29) becomes

\[ h(r, t) = \frac{1}{v(r)} S \left( t - \int_0^r \frac{dv'}{v'} \right). \]

In general at \( r = 0 \) we have

\[ S(t) = v(r = 0, t) h(r = 0, t), \]

so that \( S \) may be interpreted as the flux of the quantity \( h \) or \( f \) into the region \( r > 0 \). This interpretation enables one to obtain the boundary condition at the interface with another region, \( r < 0 \) in this case, where another transport equation applies.

### 3. Solutions in Three Dimensions

In three dimensions the conservation equation is

\[ \frac{\partial f}{\partial t} + \nabla \cdot (vf) = S(r, t). \]  

A trajectory runs from \( r_0 = (x_{01}, x_{02}, x_{03}) \) to \( r = (x_1, x_2, x_3) \) and is determined by the equation

\[ \frac{\partial r(r_0, t)}{\partial t}|_{r_0} = v(r, t). \]

The equations analogous to equation (6) which give \( r_0(r, t) \) directly are

\[ \frac{\partial x_{0i}}{\partial t} = -\frac{\partial x_{0i}}{\partial x_j} \frac{\partial x_j}{\partial t}, \]

or, in terms of a matrix \( J \) defined by,

\[ J_{ij} = \frac{\partial x_{0i}}{\partial x_j}, \]

by the equation

\[ \frac{\partial r_0}{\partial t} = -J \cdot v. \]

The important quantity determining the solution for \( f \) is the determinant of \( J \) or Jacobian of the transformation from \( r_0 \) to \( r \),

\[ J(r_0, r) = \det J = \frac{\partial (x_{01}, x_{02}, x_{03})}{\partial (x_1, x_2, x_3)}. \]

Then the solution of (32) without the source term is given, for example by Lamb (1932), as

\[ f(r, t) = f_0(r_0(r, t)) J(r_0(r, t), r). \]

By using equation (36) and the properties of the inverse matrix \( J^{-1} \), it is possible to prove directly that the solution (38) satisfies equation (32).

Of course the main problem is the determination of the trajectories, given \( v(r, t) \). Types of velocity field are classified by Truesdale & Toupin (1960) but we shall not enter this general problem.
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When the source term is included we consider a trajectory that passes through the points \((\mathbf{r}_0, 0), (\mathbf{r}_1, t_1), \) and \((\mathbf{r}, t)\). If we collect the contributions from along the path, the solution of (32) arising from the source term is

\[
h(\mathbf{r}, t) = \int_0^t S(\mathbf{r}_1(\mathbf{r}_0(\mathbf{r}, t), t_1), t_1) J(\mathbf{r}_1(\mathbf{r}_0(\mathbf{r}, t), t_1), \mathbf{r}) \, dt_1
\]

\[
= J(\mathbf{r}_0, \mathbf{r}) \int_0^t S(\mathbf{r}_1(\mathbf{r}_0(\mathbf{r}, t), t_1), t_1) J(\mathbf{r}_1, \mathbf{r}_0) \, dt_1,
\]

(39)

where we have used the determinental identity

\[
J(\mathbf{r}_1, \mathbf{r}) = J(\mathbf{r}_1, \mathbf{r}_0) J(\mathbf{r}_0, \mathbf{r}).
\]

(40)

Again it may be proved directly that (39) satisfies equation (32) with the help of the equation satisfied by \(J(\mathbf{r}_0, \mathbf{r})\),

\[
\frac{\partial J}{\partial t} + \nabla \cdot (\mathbf{v}(\mathbf{r}, t) J) = 0.
\]

(41)

The integral in (39) can be transformed to one along a path in space which is at every point directed along \(\mathbf{v}(\mathbf{r}_1, t_1(\mathbf{r}_0(\mathbf{r}, t), \mathbf{r}_1))\).

\[
h(\mathbf{r}, t) = J(\mathbf{r}_0, \mathbf{r}) \int_{\mathbf{r}_0(\mathbf{r}, t)}^{\mathbf{r}_1} S(\mathbf{r}_1, t_1(\mathbf{r}_0(\mathbf{r}, t), \mathbf{r}_1)) J(\mathbf{r}_1, \mathbf{r}_0) \frac{ds}{v(\mathbf{r}_1, t_1)},
\]

(42)

where \(ds = \mathbf{v} \cdot d\mathbf{r}_1/v\) is an element of the path.

Knowing the trajectory from a boundary point \((\mathbf{r}, t_1)\) through \((\mathbf{r}', t')\) and \((\mathbf{r}, t)\), we can write down the solution arising from the boundary,

\[
h(\mathbf{r}, t) = \int_{t_1}^{t'} S(\mathbf{r}'(\mathbf{r}, t), t_1), t') J(\mathbf{r}'(\mathbf{r}, t), t_1) \, dt'.
\]

(43)

The above discussion, with appropriate obvious modifications, applies in \(n\) dimensions. In particular, with an appropriate definition of \(\mathbf{v}\), it applies to the two-dimensional case where one dimension is spatial and the other represents a particle size. There should be practical applications of this.

4. Nonlinear equations

So far we have considered velocities which are independent of the function \(f\) to be determined. When this is not the case the general solutions obtained hitherto, such as (7), (18), (38) and (39) generally break down. If \(v\) is a function of \(f\) there is a mathematical description of the more general trajectories which arise. We consider this situation first and give a concrete physical example which illustrates a method of solution and shows that the previous solutions are incorrect. The interesting case when \(v\) depends on the moments of \(f\) is discussed finally. These moments may be regarded as functions of time only, so that the previous general solutions are still formally correct. However, the quantities such as \(\mathbf{r}_0, \mathbf{r}_1, r_1\) and \(r_1\) which are determined by the trajectories are no longer independent of \(f\). Methods are developed to tackle some particular cases of interest. In the following discussion only one space dimension will be covered although many of the results can be generalized.
(a) **Quasilinear equations**

When \( v(r, t, f) \) depends only on the function \( f \) and not on its derivatives, the general equation (3) falls into the class of quasilinear equations covered in ch. 2 of Courant & Hilbert (1962). We briefly quote some of their general results; a fuller discussion is given in the book. We can rewrite equation (3) in the form

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \left( v + f \frac{\partial v}{\partial f} \right) = S - f \frac{\partial v}{\partial x}.
\] (44)

The equation is now in the canonical form of Courant & Hilbert (1962) and the trajectories or characteristic curves in \( (t, x, f) \) space are defined in terms of a parameter \( s \) by

\[
\begin{align*}
\frac{dt}{ds} &= 1, \\
\frac{dx}{ds} &= v + f \frac{\partial v}{\partial f}, \\
\frac{df}{ds} &= S - f \frac{\partial v}{\partial x}.
\end{align*}
\] (45)

The problem of solving equation (3) is reduced to that of solving these three equations with initial conditions specified on a surface in the space. In our physical situation this surface will generally be either the plane \( t = 0 \) or a surface, \( x = \) constant. The general equations of Courant & Hilbert (1962) do not have specific solutions of the type we have obtained hitherto because they cannot be transformed back into an equation of the form (3). Evidently equations (45) suggest that the parameter \( s \) generally represents the time taken along a trajectory. It may easily be verified that some special cases we have considered are obtained from equations (45) by putting \( \frac{\partial v}{\partial f} = 0 \) and \( S = 0 \).

(b) **Application to an equation for porosity migration**

We now give a physical example of where equations (45) can be solved and where no exact solution was found until this technique was used. The transport equation concerns the movement of porosity with the pickup of fission gas in the cylindrical geometry of a nuclear fuel element (Clement & Finnis 1977). There is no source term and the velocity and function to be determined are defined by

\[
\begin{align*}
Q &= \frac{P}{P_0}, \\
v(r, t, Q) &= \left( t / t_s \right) v_E(r) Q.
\end{align*}
\] (46)

where \( P \) is the porosity with an initial value \( P_0 \), \( v_E(r) \) is a function of \( r \) only, and the expression for the velocity is valid for times greater than \( t_s \).

The velocity is directed towards \( r = 0 \) and, with the transfer of the time dependence of \( v \) to the left hand side, the transport equation becomes

\[
\frac{t}{t_s} \frac{\partial Q}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( rv_E(r) Q^2 \right).
\] (47)
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The equations (45) for a characteristic curve are

\[
\begin{align*}
\frac{dt}{ds} &= t, \\
\frac{dr}{ds} &= -2Qv_E(r), \\
\frac{dQ}{ds} &= \frac{Q^2}{r} \left( v_E + r \frac{dv_E}{dr} \right).
\end{align*}
\] (48a) (48b) (48c)

The integral of (48a) is

\[ s = t_s \ln \left( \frac{t}{t_s} \right), \] (49)

where the time is measured from \( t = t_s \).

From equations (48b) and (48c) along a trajectory we have

\[ \frac{dQ}{dr} = -\frac{Q}{2rv_E(r)} \left( v_E + r \frac{dv_E}{dr} \right), \] (50)

which has the solution,

\[ Q = Q_0(r_0) \left[ \frac{r_0 v_E(r_0)}{rv_E(r)} \right]^{\frac{1}{2}}, \] (51)

where \( r = r_0 \) initially, and \( Q_0(r_0) \) is constant and equal to unity in the original problem.

We note that \( r \leq r_0 \) since the sign is negative in (48b). By substituting (51) in (48b) we find that

\[ s = -\frac{1}{2Q_0(r_0)} \int_{r_0}^{r} dr' \left[ \frac{r'}{r_0 v_E(r_0) v_E(r')} \right]^{\frac{1}{2}}, \] (52)

so that, from (49), the relation between \( r \) and \( t \) along a trajectory is

\[ t = t_s \exp \left( -\frac{1}{2Q_0(t_s)} \int_{r_0}^{r} dr' \left[ \frac{r'}{r_0 v_E(r_0) v_E(r')} \right]^{\frac{1}{2}} \right). \] (53)

A complete solution to equation (47) is provided by equation (51) where \( r_0 \) is specified as a function of \( r \) and \( t \) by equation (53). We notice that \( r_0(r, t) \) does not satisfy equation (6) but is here a solution of the equation

\[ \frac{\partial r_0}{\partial t} = 2v(r, t, Q) \frac{\partial r_0}{\partial r} = 2\frac{t_s}{t} Q v_E(r) \frac{\partial r_0}{\partial r}. \] (54)

Also it is easy to see that \( Q \) does not satisfy equation (7) in terms of \( Q_0(r_0) \) and \( \frac{\partial r_0}{\partial r} \).

(c) Moments of distributions

Now we return to equation (2) which is the form most commonly encountered in growth problems. We assume that \( \cdot \) or \( v \) is only a function of time through moments of the distribution. For definiteness we only consider a function of the first moment, and no source term, which represents one of the situations in void swelling (Brailsford & Bullough 1972). Then \( v = v(\bar{r}(t), r) \), where

\[ \bar{r}(t) = \int dr \, r n(r, t), \] (55)

and the unspecified limits are usually 0 or some minimum value and infinity.
As has been stated previously, the formal solution (7) is still valid even though \(v_0\) now depends on \(f\). By substitution in (55) we have

\[
\tilde{r}(r) = \int dr_0 n_0 (r_0, t) \frac{\partial r_0}{\partial r} (r, t) r
= \int dr_0 n_0 (r_0) r (r_0, t),
\]

(56)

where the integration variable is changed to \(r_0\).

Together with equation (6) or equation (4), which we rewrite as

\[
\frac{\partial r(r_0, t)}{\partial t} = v(r, \tilde{r}(t)),
\]

(57)
equation (56) determines the solution from the initial distribution, \(n_0\). We note that the range of integration in (56) may not run over all \(r_0\) but, if \(v\) is negative, may be restricted to \(r_0 \geq r_{\text{omin}}(t)\) such that \(r(r_0, t) \geq 0\). In this case \(N\), defined as

\[
N(t) = \int dr \, n(r, t)
= \int_{r_{\text{omin}}(t)}^{\infty} dr_0 n_0 (r_0),
\]

(58)
is decreasing with time.

Where \(v(r, \tilde{r})\) is separable,

\[
v(r, \tilde{r}) = v(r) g(\tilde{r}(t)),
\]

(59)
the solution of equation (57) is

\[
\int_{r_0}^{r} \frac{dr'}{v(r')} = \int_{0}^{t} g(\tilde{r}(t')) \, dt' = h(t), \quad \text{say.}
\]

(60)

On substitution into (56), the equation becomes an integral equation which determines the timescale. The functional form of the solution for \(n(r, t)\) is fixed from \(n_0(r)\) and \(v(r)\) by equations (13) and (14) which take the form

\[
n(r, t) = \frac{1}{v(r)} u \left( h(t) - \int_{r_0}^{r} \frac{dr'}{v(r')} \right),
\]

(61)

\[
u \left( - \int_{r_0}^{r} \frac{dr'}{v(r')} \right) = v(r) n_0 (r).
\]

(62)

As an example, for large voids

\[
v(r) = \frac{1}{r}, \quad g(\tilde{r}) = \frac{C}{(\tilde{r} + a)(\tilde{r} + b)},
\]

(63)

where \(a, b,\) and \(C\) are constants, so that (60) gives

\[
r^2 - r_0^2 = 2 \int_{0}^{t} \frac{C}{(\tilde{r} + a)(\tilde{r} + b)} \, dt' = 2h(t).
\]

(64)
The solution is

\[
n(r, t) = \frac{r}{[r^2 - 2h(t)]^{1/2}} n_0 ([r^2 - 2h(t)]^{1/2}),
\]

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where \( h(t) \) is determined from the integral equation

\[
\tilde{r}(t) = \int dr_0 n_0(r_0) [r_0^2 + 2h(t)]^{1/2}.
\]  

(66)

Where the distribution is a \( \delta \)-function,

\[
n_0(r_0) = \delta(r_0 - \tilde{r}_0),
\]  

(67)

equation (66) gives

\[
\tilde{r}^2 = \tilde{r}_0 + 2h(t).
\]  

(68)

By differentiation we obtain the expected result,

\[
\frac{d\tilde{r}}{dt} = \frac{C}{\tilde{r}(\tilde{r} + a)(\tilde{r} + b)}.
\]  

(69)

This discussion leads us naturally to the final topic which is the behaviour of distributions in terms of their moments. From an experimental point of view this behaviour is very important since initial distributions will rarely be given exactly and might well be represented adequately by just a few numbers giving their lowest moments. We illustrate a possible approach to the problem by considering the first moment given by equation (56).

Suppose \( \tilde{r} \) has an initial value \( \tilde{r}_0 \) so that we can expand \( r(r_0, t) \) in a Taylor series as

\[
r(r_0, t) = r(\tilde{r}_0, t) + (r_0 - \tilde{r}_0) \frac{\partial r}{\partial r_0} (\tilde{r}_0, t) + \frac{1}{2} (r_0 - \tilde{r}_0)^2 \frac{\partial^2 r}{\partial r_0^2} (\tilde{r}_0, t) + \ldots
\]  

(70)

Then we can evaluate (56) in terms of the moments of the original distribution,

\[
\tilde{r}(t) = r(\tilde{r}_0, t) + \frac{1}{2} \frac{\partial^2 r}{\partial r_0^2} (\tilde{r}_0, t) \text{var} \ n_0 + \ldots,
\]  

(71)

where

\[
\text{var} \ n_0 = \int n_0(r_0) (r_0 - \tilde{r}_0)^2 \, dr_0.
\]  

(72)

The contribution from the second term in (70) vanishes from the definition of \( \tilde{r}_0 \).

The expansion (71) can be substituted into equation (57) when the first term gives the equation of motion for a \( \delta \)-function distribution at \( \tilde{r}_0 \). Higher order terms can easily be obtained by iteration when, as in the above large-void example, equation (57) can be solved. Thus an expansion procedure emerges for solving equations (56) and (57) which gives corrections to a \( \delta \)-function approximation for the original distribution. If the results of this expansion converge slowly, or even diverge, it implies that an initial physical distribution must be very well determined for it to be calculated accurately at later times.

The expansion procedure can be generalized to deal with higher moments and also with a two-peaked distribution. For example, one peak, \( \tilde{r}_1 \), might be in a region where \( \nu \) is positive and another peak, \( \tilde{r}_2 \), where \( \nu \) is negative. The equations (57) to the lowest order, would then consist of two ordinary coupled differential equations for \( \tilde{r}_1 \) and \( \tilde{r}_2 \).
5. DISCUSSION AND CONCLUSIONS

A number of practical methods have been given for the solution of the continuity equation. Explicit solutions are obtainable when \( v \) or \( \dot{r} \) is separable in \( r \) and \( t \) which, in the author's experience, appears to be quite common in applications. A final practical extension of the previous work, where this separability may often be applied, is the use of perturbation theory.

Let us define \( v = v_1 + v_2 \), where \( v_1 \gg v_2 \), and \( f = f_1 + f_2 \) such that \( f_1 \) satisfies

\[
\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial r} (f_1 v_1) = 0.
\]

(73)

and

\[
f_1(r, 0) = f_0(r).
\]

(74)

Then \( f_2 \) satisfies the equation

\[
\frac{\partial f_2}{\partial t} + \frac{\partial}{\partial r} [f_2 (v_1 + v_2)] = -\frac{\partial}{\partial r} [f_1 v_2].
\]

(75)

Now, if we can neglect the term arising from \( f_2 v_2 \) on the left hand side, the solutions required for \( f_2 \) are precisely those given in §2b with the motion defined by \( v_1 \) and a source term,

\[
S(r, t) = -\frac{\partial}{\partial r} [f_1 v_2].
\]

(76)

In a more speculative application of perturbation theory we can treat the equation with a small diffusive term,

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial r} (fv) = D \frac{\partial^2 f}{\partial r^2},
\]

(77)

where \( D \) is the diffusivity.

Then we again obtain equation (75) for \( f_2 \) with a source term

\[
S(r, t) = D \frac{\partial^2 f_1}{\partial r^2}.
\]

(78)

One must be very careful in applying perturbation theory to a diffusive term. First the solution will obviously break down if the second derivatives become large enough. Secondly there is the influence of boundary conditions. With a general boundary condition the original solution may only dominate in a region to which a diffusive contribution has not had time to propagate from the boundary. In the special case where \( v \) and \( \frac{\partial f}{\partial r} \) vanish at the boundaries so that transport is entirely internal, perturbation theory will have a wider range of applicability.

The formal solution (7) can be used as a starting point for the examination of continuity equations containing terms representing the Smoluchowski coagulation of particles (for a brief summary of this subject see Dunning (1973b)).

\[
\frac{\partial n}{\partial t} + \frac{\partial}{\partial r} (nv) = \frac{1}{2} \int_0^r \beta(r', r - r') n(r', t) n(r - r', t) \, dr' - \int_0^\infty \beta(r, r') n(r', t) n(r, t) \, dr.
\]

(79)

The complete solution may be written as

\[
n(r, t) = n_0(r_0(r, t)) \frac{\partial r_0}{\partial r} (r, t) + h(r, t),
\]

(80)
where the first term is the solution of the equation with no coagulation or 'collision' terms.

An integral equation for \( h(r, t) \) is obtained when this solution is substituted into the right hand side of equation (79) which is regarded as a source term for that equation with the solution \( h(r, t) \). The transformation of equation (79) into an integral equation may make it easier to handle.

In conclusion we have written down the formal solution of the continuity equation when \( v \) is a function of \( r \) and \( t \), only. Explicit solutions have been obtained to the equation with source terms in space and time. The corresponding solutions in the three-dimensional case have been briefly considered. Where \( v \) is also a function of \( f \) a general method of solution of the equation has been presented together with a physical non-trivial example. When \( v \) depends on the moments of \( f \), some specific results have been obtained, and a fairly general method has been devised to investigate the behaviour of distributions in terms of initially specified moments. Some extensions of the theory to more general problems have been mentioned. There is scope for further theoretical work on the partial integro-differential equations involving moments of distributions, especially where boundary conditions are involved (Ostwald ripening), and the coagulation terms which also occur in nucleation theory.

The potential for the application of the present results to physics and chemistry, and indeed any subject where a continuity equation applies, is very large. Applications have been mentioned to porosity motion in nuclear fuel elements and void swelling in steels. Although the basic solutions (7) and (37) have been known for a long time in some parts of physics, they have not previously been applied to these problems. It is the author's hope that this paper will enable more applications to be made to practical problems.

References