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# RAYLEIGH SCATTERING IN PLANETARY ATMOSPHERES: CORRECTED TABLES THROUGH ACCURATE COMPUTATION OF X AND Y FUNCTIONS

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## ABSTRACT

Tables that have been used as a reference for nearly 50 years for the intensity and polarization of reflected and transmitted light in Rayleigh scattering atmospheres have been found to be inaccurate, even to four decimal places. We convert the integral equations describing the *X* and *Y* functions into a pair of coupled integro-differential equations that can be efficiently solved numerically. Special care has been taken in evaluating Cauchy principal value integrals and their derivatives that appear in the solution of the Rayleigh scattering problem. The new approach gives results accurate to eight decimal places for the entire range of tabulation (optical thicknesses 0.02–1.0, surface reflectances 0–0.8, solar and viewing zenith angles 0°–88.85°, and relative azimuth angles 0°–180°), including the most difficult case of direct transmission in the direction of the sun. Revised tables have been created and stored electronically for easy reference by the planetary science and astrophysics community.

*Key words:* polarization – radiative transfer

## 1. INTRODUCTION

The radiative transfer equations describing the intensity and polarization of light reflected and transmitted by a Rayleigh scattering atmosphere are well known and were solved by Chandrasekhar (1960). On the basis of these solutions, Coulson et al. (1960) generated tables of quantities describing the radiation emerging from such an atmosphere. These tables have been used as a reference standard for nearly 50 years for checking the veracity of radiative transfer codes and as look-up tables to account for Rayleigh scattering in planetary atmospheres. However, we recently found that the tables were not accurate to one or two units in the fifth decimal place, despite the claim by Coulson et al. (1960). This is a very serious problem as polarimetry has emerged as an active area of research in astronomy (see, e.g., Stam et al. 2004; Bailey 2007; Berdyugina et al. 2008), remote sensing of $CO_2$ (Natraj et al. 2007, 2008), and $O_3$ in the terrestrial atmosphere (Guo et al. 2007). Measurements of polarization can now be made to a precision of $10^{-6}$ (Hough et al. 2006). Therefore, modeling of polarization must exceed the same level of accuracy.

In this paper, we examine the problem of Rayleigh scattering from a new perspective, analyze the reasons for the low precision of the previous tabulations, and present higher precision results for future reference. Section 2 describes the basic theory of Rayleigh scattering and introduces the *X* and *Y* functions that are central to the problem. The computation of the *X* and *Y* functions and their derivatives is discussed in Section 3. Some sample results are presented in Section 4.

We follow the notation of Chandrasekhar (1960), which is considered elegant and classic. In addition, since we are comparing our results to those compiled in the tables of Coulson et al. (1960), it is important that we use the same notation as in the tables to minimize confusion. To facilitate comparison with later work that uses alternative notations, we include information on how to convert between them.

## 2. BASIC THEORY

Polarized light can be completely described by four quantities (Stokes 1852; Chandrasekhar 1960) *I*, *Q*, *U*, and *V*, where *I* is the intensity, *Q* and *U* describe the linearly polarized radiation, and *V* refers to the circularly polarized radiation. These quantities are called Stokes parameters. In the following discussion, we first present expressions to compute *I* and *Q*, and then those for *U*. *V* is not discussed further because it is identically zero for a Rayleigh scattering atmosphere when the incoming light is unpolarized (as is the case for sunlight).

*I* and *Q* can further be expressed in terms of radiation in two arbitrary directions at right angles to one another in the plane transverse to the direction of propagation of the radiation. If we call these directions *l* (parallel) and *r* (perpendicular), then the following expressions hold:

$$I = I_l + I_r \tag{1a}$$

$$Q = I_r - I_l. \tag{1b}$$

Note that the literature is split on the definition of *Q*. Some (e.g., Hansen & Travis 1974) use the opposite sign convention. Here we use the definition given in Coulson et al. (1960).

For a Rayleigh scattering atmosphere bounded by a dark surface (surface reflectance = 0), $I_l$ and $I_r$ can be written in the form (Coulson et al. 1960)

$$I_l(\tau; \mu, \varphi) = I_l^0(\tau; \mu, \mu_0) + I_l^1(\tau; \mu, \mu_0)\cos\varphi + I_l^2(\tau; \mu, \mu_0)\cos 2\varphi \tag{2a}$$

$$I_r(\tau; \mu, \varphi) = I_r^0(\tau; \mu, \mu_0) + I_r^1(\tau; \mu, \mu_0)\cos\varphi + I_r^2(\tau; \mu, \mu_0)\cos 2\varphi, \tag{2b}$$

where $\tau$, $\mu$, and $\varphi$ denote the optical thickness (measured downward from the upper boundary), the cosine of the polar angle (with respect to the outward normal), and the azimuthal angle (measured counterclockwise, looking down, from an arbitrary but fixed direction), respectively. For a horizontally homogeneous atmosphere, only the relative azimuth angle between the outgoing and incident directions is relevant. In this paper, the azimuth angle of incident sunlight is taken to be zero. Subscript "0" refers to the incident direction.

The following expressions hold for the zeroth Fourier components:

$$I_l^0(\tau; -\mu, -\mu_0) = \frac{C}{\mu - \mu_0}[K\xi(\mu) + L\eta(\mu) - M\psi(\mu) - N\phi(\mu)] \tag{3a}$$

$$I_r^0(\tau; -\mu, -\mu_0) = \frac{C}{\mu - \mu_0}[K\sigma(\mu) + L\theta(\mu) - M\chi(\mu) - N\zeta(\mu)] \tag{3b}$$

$$I_l^0(0; \mu, -\mu_0) = \frac{C}{\mu + \mu_0}[K\psi(\mu) + L\phi(\mu) - M\xi(\mu) - N\eta(\mu)] \tag{3c}$$

$$I_r^0(0; \mu, -\mu_0) = \frac{C}{\mu + \mu_0}[K\chi(\mu) + L\zeta(\mu) - M\sigma(\mu) - N\theta(\mu)], \tag{3d}$$

where the first two equations refer to diffuse downwelling radiation emerging from the bottom of the atmosphere (BOA) and the latter two are for the diffuse upwelling radiation at the top of the atmosphere (TOA). Note that Hansen & Travis (1974) use $b$ to denote the optical thickness of the atmosphere.

The constant C is given by

$$C = \tfrac{3}{32}\mu_0 F_0, \tag{4}$$

where $\pi F_0$ is the flux of incident radiation (measured perpendicular to the direction of incidence) at the TOA. The depolarization factor for Rayleigh scattering is assumed to be zero, to be consistent with Coulson et al. (1960).

$K$, $L$, $M$, and $N$ can be calculated as follows:

$$K = \psi(\mu_0) + \chi(\mu_0) \tag{5a}$$

$$L = 2[\phi(\mu_0) + \zeta(\mu_0)] \tag{5b}$$

$$M = \xi(\mu_0) + \sigma(\mu_0) \tag{5c}$$

$$N = 2[\eta(\mu_0) + \theta(\mu_0)]. \tag{5d}$$

The functions $\psi, \phi, \xi, \eta, \chi, \zeta, \sigma$, and $\theta$ are linear combinations of the pairs of $X$ and $Y$ functions $(X_l, Y_l)$ and $(X_r, Y_r)$ (Chandrasekhar & Elbert 1954)

$$\psi(\mu) = \mu[\nu_1 Y_l(\mu) - \nu_2 X_l(\mu)], \tag{6a}$$

$$\phi(\mu) = (1 + \nu_4\mu)X_l(\mu) - \nu_3\mu Y_l(\mu); \tag{6b}$$

$$\xi(\mu) = \mu[\nu_2 Y_l(\mu) - \nu_1 X_l(\mu)]; \tag{6c}$$

$$\eta(\mu) = (1 - \nu_4\mu)Y_l(\mu) + \nu_3\mu X_l(\mu); \tag{6d}$$

$$\chi(\mu) = (1 - u_4\mu)X_r(\mu) + u_3\mu Y_r(\mu) + Q(u_4 - u_3)\mu^2[X_r(\mu) - Y_r(\mu)]; \tag{6e}$$

$$\zeta(\mu) = \tfrac{1}{2}\mu[\nu_1 Y_r(\mu) - \nu_2 X_r(\mu)] + \tfrac{1}{2}Q(\nu_2 - \nu_1)\mu^2[X_r(\mu) - Y_r(\mu)]; \tag{6f}$$

$$\sigma(\mu) = (1 + u_4\mu)Y_r(\mu) - u_3\mu X_r(\mu) - Q(u_4 - u_3)\mu^2[X_r(\mu) - Y_r(\mu)]; \tag{6g}$$

$$\theta(\mu) = \tfrac{1}{2}\mu[\nu_2 Y_r(\mu) - \nu_1 X_r(\mu)] - \tfrac{1}{2}Q(\nu_2 - \nu_1)\mu^2[X_r(\mu) - Y_r(\mu)]. \tag{6h}$$

The unknowns in Equations (6) can be calculated using the following expressions:[1]

$$\nu_2 + \nu_1 = 2\Delta_1(\kappa_1\delta_1 - \kappa_2\delta_2) \tag{7a}$$

$$\nu_2 - \nu_1 = 2\Delta_2(\kappa_1\delta_1 - \kappa_2\delta_2) \tag{7b}$$

$$\nu_4 + \nu_3 = \Delta_1(d_1\kappa_1 - d_0\kappa_2) \tag{7c}$$

$$\nu_4 - \nu_3 = \Delta_2\,[c_1\delta_1 - c_0\delta_2 - 2Q(d_0\delta_1 - d_1\delta_2)] \tag{7d}$$

$$u_4 + u_3 = \Delta_1(c_1\delta_1 - c_0\delta_2) \tag{7e}$$

$$u_4 - u_3 = \Delta_2(d_1\kappa_1 - d_0\kappa_2) \tag{7f}$$

$$Q = \frac{c_0 - c_2}{(d_0 - d_2)\tau + 2(d_1 - d_3)}, \tag{7g}$$

where $\Delta_1$, $\Delta_2$, $c_n$, $d_n$, $\kappa_n$, and $\delta_n$ are given by

$$\Delta_1 = \frac{1}{d_0\delta_1 - d_1\delta_2} \tag{8a}$$

$$\Delta_2 = \frac{1}{c_0\kappa_1 - c_1\kappa_2 - 2Q(d_1\kappa_1 - d_0\kappa_2)} \tag{8b}$$

$$c_0 = A_0 + B_0 - \frac{8}{3} \tag{8c}$$

$$d_0 = A_0 - B_0 - \frac{8}{3} \tag{8d}$$

$$c_n = A_n + B_n \qquad (n = 1,\ 2) \tag{8e}$$

$$d_n = A_n - B_n \qquad (n = 1,\ 2,\ 3) \tag{8f}$$

$$\kappa_n = \alpha_n + \beta_n \qquad (n = 1,\ 2) \tag{8g}$$

$$\delta_n = \alpha_n - \beta_n \qquad (n = 1,\ 2). \tag{8h}$$

$A_n$, $B_n$, $\alpha_n$, and $\beta_n$ are moments of order $n$ of $X_r$, $Y_r$, $X_l$, and $Y_l$, respectively. The constants in Equations (7) and (8) depend only on the optical thickness.

The Chandrasekhar $X$ and $Y$ functions (Chandrasekhar 1960) in general satisfy the simultaneous nonlinear integral equations

$$X(\mu) = 1 + \mu\int_0^1 \Psi(\mu')\left[\frac{X(\mu)X(\mu') - Y(\mu)Y(\mu')}{\mu + \mu'}\right]d\mu' \tag{9a}$$

$$Y(\mu) = e^{-\tau/\mu} + \mu\int_0^1 \Psi(\mu')\left[\frac{Y(\mu)X(\mu') - X(\mu)Y(\mu')}{\mu - \mu'}\right]d\mu'. \tag{9b}$$

$\Psi$ is called the characteristic function. $(X_l, Y_l)$ and $(X_r, Y_r)$ have the characteristic functions $\Psi_l$ and $\Psi_r$, respectively

$$\Psi_l(\mu) = \tfrac{3}{4}(1 - \mu^2); \tag{10a}$$

$$\Psi_r(\mu) = \tfrac{3}{8}(1 - \mu^2). \tag{10b}$$

[1] The constant $Q$ is to be distinguished from Stokes parameter $Q$.

The first and second Fourier components can be computed as follows:

$$I_l^1(\tau;-\mu,-\mu_0)=\frac{4C}{\mu-\mu_0}\mu\mu_0(1-\mu^2)^{1/2}\left(1-\mu_0^2\right)^{1/2}\times W^1(\mu,\mu_0) \quad (11a)$$

$$I_r^1(\tau;-\mu,-\mu_0)=0 \quad (11b)$$

$$I_l^1(0;\mu,-\mu_0)=-\frac{4C}{\mu+\mu_0}\mu\mu_0(1-\mu^2)^{1/2}\left(1-\mu_0^2\right)^{1/2}\times M^1(\mu,\mu_0) \quad (11c)$$

$$I_r^1(0;\mu,-\mu_0)=0 \quad (11d)$$

$$I_l^2(\tau;-\mu,-\mu_0)=-\frac{C}{\mu-\mu_0}\mu^2\left(1-\mu_0^2\right)W^2(\mu,\mu_0) \quad (11e)$$

$$I_r^2(\tau;-\mu,-\mu_0)=\frac{C}{\mu-\mu_0}\left(1-\mu_0^2\right)W^2(\mu,\mu_0) \quad (11f)$$

$$I_l^2(0;\mu,-\mu_0)=-\frac{C}{\mu+\mu_0}\mu^2\left(1-\mu_0^2\right)M^2(\mu,\mu_0) \quad (11g)$$

$$I_r^2(0;\mu,-\mu_0)=\frac{C}{\mu+\mu_0}\left(1-\mu_0^2\right)M^2(\mu,\mu_0), \quad (11h)$$

where

$$W^n(\mu,\mu_0)=X^n(\mu_0)Y^n(\mu)-Y^n(\mu_0)X^n(\mu) \qquad (n=1,\ 2); \quad (12a)$$

$$M^n(\mu,\mu_0)=X^n(\mu_0)X^n(\mu)-Y^n(\mu_0)Y^n(\mu) \qquad (n=1,\ 2). \quad (12b)$$

$(X^1, Y^1)$ and $(X^2, Y^2)$ have the characteristic functions $\Psi^1$ and $\Psi^2$, respectively:

$$\Psi^1(\mu)=\tfrac{3}{8}(1-\mu^2)(1+2\mu^2) \quad (13a)$$

$$\Psi^2(\mu)=\tfrac{3}{16}(1+\mu^2)^2. \quad (13b)$$

Stokes parameter $U$ is given by the expression

$$U(\tau;\mu,\varphi)=U^1(\tau;\mu,\mu_0)\sin\varphi+U^2(\tau;\mu,\mu_0)\sin 2\varphi, \quad (14)$$

where

$$U^1(\tau;-\mu,-\mu_0)=\frac{4C}{\mu-\mu_0}\mu_0(1-\mu^2)^{1/2}\left(1-\mu_0^2\right)^{1/2}W^1(\mu,\mu_0) \quad (15a)$$

$$U^1(0;\mu,-\mu_0)=\frac{4C}{\mu+\mu_0}\mu_0(1-\mu^2)^{1/2}\left(1-\mu_0^2\right)^{1/2}M^1(\mu,\mu_0) \quad (15b)$$

$$U^2(\tau;-\mu,-\mu_0)=-\frac{2C}{\mu-\mu_0}\mu\left(1-\mu_0^2\right)W^2(\mu,\mu_0) \quad (15c)$$

$$U^2(0;\mu,-\mu_0)=\frac{2C}{\mu+\mu_0}\mu\left(1-\mu_0^2\right)M^2(\mu,\mu_0). \quad (15d)$$

For the case of nonzero surface reflection, additional terms are necessary to compute the Stokes parameters. If the surface is Lambertian, Stokes parameter $U$ is unaffected. $I_l$ and $I_r$ can be expressed as follows:

$$I_l(\tau;\mu,\varphi)=I_l^0(\tau;\mu,\mu_0)+I_l^1(\tau;\mu,\mu_0)\cos\varphi + I_l^2(\tau;\mu,\mu_0)\cos 2\varphi+I_l^*(\tau;\mu,\mu_0) \quad (16a)$$

$$I_r(\tau;\mu,\varphi)=I_r^0(\tau;\mu,\mu_0)+I_r^1(\tau;\mu,\mu_0)\cos\varphi + I_r^2(\tau;\mu,\mu_0)\cos 2\varphi+I_r^*(\tau;\mu,\mu_0). \quad (16b)$$

The additional terms $I_l^*$ and $I_r^*$ (due to the effect of nonzero surface reflectance) can be described by the following simple expressions:

$$I_l^*(\tau;-\mu,-\mu_0)=H[1-\gamma_l(\mu)] \quad (17a)$$

$$I_r^*(\tau;-\mu,-\mu_0)=H[1-\gamma_r(\mu)] \quad (17b)$$

$$I_l^*(0;\mu,-\mu_0)=H\gamma_l(\mu) \quad (17c)$$

$$I_r^*(0;\mu,-\mu_0)=H\gamma_r(\mu), \quad (17d)$$

where

$$H=\frac{A\mu_0F_0}{4(1-A\bar{s})}[\gamma_l(\mu_0)+\gamma_r(\mu_0)]; \quad (18a)$$

$$\gamma_l(\mu)=\frac{3}{8}Q(\nu_2-\nu_1)(d_0-d_2)[X_l(\mu)+Y_l(\mu)]; \quad (18b)$$

$$\gamma_r(\mu)=\frac{3}{8}Q(d_0-d_2)[(u_4-u_3)\{X_r(\mu)+Y_r(\mu)\} - u_5\mu\{X_r(\mu)-Y_r(\mu)\}]. \quad (18c)$$

$A$ is the surface reflectance. The quantities $\bar{s}$ and $u_5$ depend only on the optical thickness

$$\bar{s}=1-\tfrac{3}{8}Q(d_0-d_2)[(\nu_2-\nu_1)\kappa_1+(u_4-u_3)c_1-u_5d_2]; \quad (19a)$$

$$u_5=\Delta_2(c_0\kappa_1-c_1\kappa_2). \quad (19b)$$

## 3. COMPUTATION OF THE $X$ AND $Y$ FUNCTIONS AND THEIR DERIVATIVES

It is clear from the above equations that the central issue in the solution of the Rayleigh scattering problem involves the computation of the various $X$ and $Y$ functions. Coulson et al. (1960) obtained the necessary quantities from tables computed by Sekera & Blanch (1952), Sekera & Ashburn (1953), and Chandrasekhar & Elbert (1954). All of the above relied on an iterative solution of Equations (9) in the integral form. However, due to the singularity introduced by the denominator of Equation (9b), there are convergence issues associated with this technique. Bellman et al. (1966) developed an alternative solution by transforming Equations (9) to a pair of coupled integro-differential equations, which are particularly suitable for numerical solution using computers.

$$\frac{\partial X(\mu)}{\partial\tau}=Y(\mu)\int_0^1\Psi(\mu')\frac{Y(\mu')}{\mu'}d\mu' \quad (20a)$$

$$\frac{\partial Y(\mu)}{\partial\tau}=-\frac{Y(\mu)}{\mu}+X(\mu)\int_0^1\Psi(\mu')\frac{Y(\mu')}{\mu'}d\mu'. \quad (20b)$$

**Table 1**
$I$ (Upwelling at TOA) for $\tau = 0.5$, $\mu_0 = 0.2$, and $A = 0.0$

| $\mu$ | $\varphi = 0°$ | $\varphi = 30°$ | $\varphi = 60°$ | $\varphi = 90°$ | $\varphi = 120°$ | $\varphi = 150°$ | $\varphi = 180°$ |
|---|---|---|---|---|---|---|---|
| 0.02 | 0.44129802 | 0.39444956 | 0.30091208 | 0.25465866 | 0.30253774 | 0.39726529 | 0.44454934 |
| 0.06 | 0.39250948 | 0.35223311 | 0.27210130 | 0.23339588 | 0.27639321 | 0.35966691 | 0.40109329 |
| 0.10 | 0.35082997 | 0.31575536 | 0.24622966 | 0.21348201 | 0.25258702 | 0.32676664 | 0.36354469 |
| 0.16 | 0.29863696 | 0.26973242 | 0.21277028 | 0.18702656 | 0.22140580 | 0.28468958 | 0.31590800 |
| 0.20 | 0.26939388 | 0.24383765 | 0.19368088 | 0.17169133 | 0.20342523 | 0.26071536 | 0.28888259 |
| 0.28 | 0.22153000 | 0.20137817 | 0.16217962 | 0.14615212 | 0.17344750 | 0.22089470 | 0.24406575 |
| 0.32 | 0.20185562 | 0.18392101 | 0.14920591 | 0.13557847 | 0.16097330 | 0.20430272 | 0.22539038 |
| 0.40 | 0.16889020 | 0.15469709 | 0.12752450 | 0.11786073 | 0.13989890 | 0.17613018 | 0.19363900 |
| 0.52 | 0.13097041 | 0.12117362 | 0.10280748 | 0.09759154 | 0.11532259 | 0.14285044 | 0.15600064 |
| 0.64 | 0.10231626 | 0.09595756 | 0.08440503 | 0.08239375 | 0.09628240 | 0.11652976 | 0.12607100 |
| 0.72 | 0.08683737 | 0.08239718 | 0.07459712 | 0.07418876 | 0.08561228 | 0.10147599 | 0.10886768 |
| 0.84 | 0.06783499 | 0.06583006 | 0.06269131 | 0.06393739 | 0.07157324 | 0.08121402 | 0.08559886 |
| 0.92 | 0.05781217 | 0.05713933 | 0.05643322 | 0.05814731 | 0.06295443 | 0.06843439 | 0.07085458 |
| 0.96 | 0.05385796 | 0.05372482 | 0.05391877 | 0.05550325 | 0.05861141 | 0.06185271 | 0.06324324 |
| 0.98 | 0.05240320 | 0.05247332 | 0.05294178 | 0.05423678 | 0.05628819 | 0.05826948 | 0.05909602 |
| 1.00 | 0.05300496 | 0.05300496 | 0.05300496 | 0.05300496 | 0.05300496 | 0.05300496 | 0.05300496 |

**Table 2**
$I$ (Downwelling at BOA) for $\tau = 0.5$, $\mu_0 = 0.2$, and $A = 0.0$

| $\mu$ | $\varphi = 0°$ | $\varphi = 30°$ | $\varphi = 60°$ | $\varphi = 90°$ | $\varphi = 120°$ | $\varphi = 150°$ | $\varphi = 180°$ |
|---|---|---|---|---|---|---|---|
| 0.02 | 0.09703171 | 0.08901596 | 0.07295573 | 0.06483276 | 0.07266281 | 0.08850861 | 0.09644588 |
| 0.06 | 0.11553675 | 0.10592439 | 0.08659640 | 0.07659862 | 0.08554343 | 0.10410059 | 0.11343081 |
| 0.10 | 0.13485741 | 0.12338692 | 0.10024223 | 0.08801144 | 0.09816504 | 0.11978912 | 0.13070302 |
| 0.16 | 0.15368433 | 0.14027168 | 0.11306987 | 0.09825212 | 0.10923106 | 0.13362266 | 0.14600671 |
| 0.20 | 0.15838610 | 0.14444914 | 0.11608879 | 0.10033646 | 0.11112912 | 0.13585873 | 0.14846675 |
| 0.28 | 0.15618358 | 0.14242837 | 0.11424823 | 0.09799362 | 0.10741978 | 0.13060115 | 0.14252669 |
| 0.32 | 0.15191358 | 0.13860786 | 0.11125496 | 0.09518205 | 0.10369486 | 0.12551338 | 0.13679338 |
| 0.40 | 0.14070011 | 0.12863410 | 0.10365367 | 0.08842135 | 0.09500316 | 0.11365098 | 0.12339910 |
| 0.52 | 0.12178394 | 0.11192848 | 0.09128777 | 0.07796226 | 0.08180742 | 0.09550803 | 0.10282324 |
| 0.64 | 0.10305234 | 0.09550874 | 0.07949285 | 0.06848327 | 0.07002363 | 0.07910758 | 0.08411390 |
| 0.72 | 0.09102071 | 0.08502244 | 0.07214297 | 0.06284951 | 0.06314035 | 0.06942945 | 0.07301547 |
| 0.84 | 0.07348081 | 0.06981390 | 0.06174753 | 0.05534670 | 0.05427833 | 0.05687686 | 0.05854240 |
| 0.92 | 0.06170619 | 0.05964952 | 0.05499023 | 0.05089607 | 0.04942372 | 0.05000805 | 0.05057318 |
| 0.96 | 0.05544795 | 0.05425758 | 0.05148138 | 0.04881521 | 0.04744944 | 0.04727404 | 0.04738406 |
| 0.98 | 0.05198745 | 0.05127723 | 0.04957393 | 0.04780805 | 0.04668981 | 0.04628180 | 0.04621923 |
| 1.00 | 0.04682203 | 0.04682203 | 0.04682203 | 0.04682203 | 0.04682203 | 0.04682203 | 0.04682203 |

**Table 3**
$Q$ (Upwelling at TOA) for $\tau = 0.5$, $\mu_0 = 0.2$, and $A = 0.0$

| $\mu$ | $\varphi = 0°$ | $\varphi = 30°$ | $\varphi = 60°$ | $\varphi = 90°$ | $\varphi = 120°$ | $\varphi = 150°$ | $\varphi = 180°$ |
|---|---|---|---|---|---|---|---|
| 0.02 | −0.01753141 | −0.06485313 | −0.15965601 | −0.20757277 | −0.16128167 | −0.06766886 | −0.02078273 |
| 0.06 | −0.01772108 | −0.05944266 | −0.14330674 | −0.18659926 | −0.14759865 | −0.06687645 | −0.02630489 |
| 0.10 | −0.01524525 | −0.05274909 | −0.12838028 | −0.16821107 | −0.13473764 | −0.06376036 | −0.02795997 |
| 0.16 | −0.00990697 | −0.04270497 | −0.10914792 | −0.14510675 | −0.11778344 | −0.05766213 | −0.02717801 |
| 0.20 | −0.00608288 | −0.03648858 | −0.09825566 | −0.13222805 | −0.10800002 | −0.05336629 | −0.02557158 |
| 0.28 | 0.00131430 | −0.02554220 | −0.08036031 | −0.11134114 | −0.09162819 | −0.04505874 | −0.02122145 |
| 0.32 | 0.00477642 | −0.02076298 | −0.07299588 | −0.10284244 | −0.08476326 | −0.04114469 | −0.01875835 |
| 0.40 | 0.01119511 | −0.01235218 | −0.06066038 | −0.08873702 | −0.07303478 | −0.03378526 | −0.01355369 |
| 0.52 | 0.01968930 | −0.00196538 | −0.04650218 | −0.07273772 | −0.05901730 | −0.02364220 | −0.00534093 |
| 0.64 | 0.02701775 | 0.00644566 | −0.03586341 | −0.06078292 | −0.04774078 | −0.01412655 | 0.00326301 |
| 0.72 | 0.03129776 | 0.01117009 | −0.03016559 | −0.05432509 | −0.04118074 | −0.00790872 | 0.00926745 |
| 0.84 | 0.03668441 | 0.01698497 | −0.02328501 | −0.04623546 | −0.03216694 | 0.00160101 | 0.01892054 |
| 0.92 | 0.03923450 | 0.01977043 | −0.01979730 | −0.04164830 | −0.02631851 | 0.00847536 | 0.02619209 |
| 0.96 | 0.03983604 | 0.02053460 | −0.01852852 | −0.03954758 | −0.02322116 | 0.01240671 | 0.03045076 |
| 0.98 | 0.03969919 | 0.02052770 | −0.01814346 | −0.03853982 | −0.02148987 | 0.01473155 | 0.03300636 |
| 1.00 | 0.03755859 | 0.01877930 | −0.01877930 | −0.03755859 | −0.01877930 | 0.01877930 | 0.03755859 |

**Table 4**
$Q$ (Downwelling at BOA) for $\tau = 0.5$, $\mu_0 = 0.2$, and $A = 0.0$

| $\mu$ | $\varphi = 0°$ | $\varphi = 30°$ | $\varphi = 60°$ | $\varphi = 90°$ | $\varphi = 120°$ | $\varphi = 150°$ | $\varphi = 180°$ |
|---|---|---|---|---|---|---|---|
| 0.02 | −0.00920833 | −0.01715198 | −0.03301054 | −0.04084698 | −0.03271763 | −0.01664463 | −0.00862250 |
| 0.06 | −0.01266323 | −0.02206189 | −0.04075593 | −0.04976917 | −0.03970296 | −0.02023809 | −0.01055729 |
| 0.10 | −0.01538451 | −0.02652451 | −0.04860080 | −0.05898050 | −0.04652360 | −0.02292670 | −0.01123012 |
| 0.16 | −0.01746551 | −0.03052730 | −0.05627439 | −0.06793108 | −0.05243558 | −0.02387828 | −0.00978788 |
| 0.20 | −0.01767297 | −0.03138703 | −0.05832873 | −0.07022742 | −0.05336906 | −0.02279663 | −0.00775363 |
| 0.28 | −0.01625676 | −0.03036695 | −0.05791762 | −0.06952842 | −0.05108917 | −0.01853974 | −0.00259987 |
| 0.32 | −0.01495157 | −0.02903636 | −0.05646445 | −0.06778205 | −0.04890435 | −0.01594187 | 0.00016863 |
| 0.40 | −0.01160882 | −0.02551200 | −0.05246997 | −0.06320685 | −0.04381947 | −0.01052889 | 0.00569219 |
| 0.52 | −0.00542392 | −0.01910281 | −0.04553080 | −0.05573965 | −0.03605045 | −0.00268237 | 0.01353678 |
| 0.64 | 0.00169861 | −0.01201444 | −0.03851183 | −0.04875891 | −0.02904261 | 0.00438673 | 0.02063705 |
| 0.72 | 0.00689487 | −0.00700779 | −0.03393019 | −0.04453766 | −0.02492757 | 0.00858520 | 0.02490011 |
| 0.84 | 0.01547851 | 0.00103248 | −0.02712704 | −0.03883915 | −0.01965783 | 0.01396952 | 0.03041692 |
| 0.92 | 0.02204865 | 0.00703628 | −0.02244251 | −0.03541740 | −0.01687601 | 0.01667775 | 0.03318166 |
| 0.96 | 0.02590733 | 0.01051101 | −0.01988620 | −0.03380673 | −0.01585426 | 0.01749454 | 0.03397123 |
| 0.98 | 0.02821492 | 0.01257041 | −0.01843576 | −0.03302461 | −0.01555164 | 0.01756584 | 0.03398315 |
| 1.00 | 0.03225729 | 0.01612864 | −0.01612864 | −0.03225729 | −0.01612864 | 0.01612864 | 0.03225729 |

**Table 5**
$U$ (Upwelling at TOA) for $\tau = 0.5$, $\mu_0 = 0.2$, and $A = 0.0$

| $\mu$ | $\varphi = 0°$ | $\varphi = 30°$ | $\varphi = 60°$ | $\varphi = 90°$ | $\varphi = 120°$ | $\varphi = 150°$ | $\varphi = 180°$ |
|---|---|---|---|---|---|---|---|
| 0.02 | 0 | 0.04390364 | 0.07365528 | 0.08128298 | 0.06713098 | 0.03737934 | 0 |
| 0.06 | 0 | 0.04428736 | 0.07046980 | 0.07153176 | 0.05342685 | 0.02724440 | 0 |
| 0.10 | 0 | 0.04435776 | 0.06762732 | 0.06357361 | 0.04248541 | 0.01921585 | 0 |
| 0.16 | 0 | 0.04408551 | 0.06384064 | 0.05397200 | 0.02964161 | 0.00988649 | 0 |
| 0.20 | 0 | 0.04374666 | 0.06158007 | 0.04872177 | 0.02280850 | 0.00497510 | 0 |
| 0.28 | 0 | 0.04291904 | 0.05764878 | 0.04024242 | 0.01205313 | −0.00267662 | 0 |
| 0.32 | 0 | 0.04248227 | 0.05594215 | 0.03677307 | 0.00775068 | −0.00570920 | 0 |
| 0.40 | 0 | 0.04161531 | 0.05293867 | 0.03093600 | 0.00064404 | −0.01067932 | 0 |
| 0.52 | 0 | 0.04036105 | 0.04917038 | 0.02406753 | −0.00748419 | −0.01629352 | 0 |
| 0.64 | 0 | 0.03913232 | 0.04592516 | 0.01855839 | −0.01378109 | −0.02057393 | 0 |
| 0.72 | 0 | 0.03828742 | 0.04388718 | 0.01529883 | −0.01738884 | −0.02298860 | 0 |
| 0.84 | 0 | 0.03686504 | 0.04073530 | 0.01057373 | −0.02242105 | −0.02629131 | 0 |
| 0.92 | 0 | 0.03563204 | 0.03822653 | 0.00708827 | −0.02594929 | −0.02854377 | 0 |
| 0.96 | 0 | 0.03475930 | 0.03654849 | 0.00488817 | −0.02808194 | −0.02987113 | 0 |
| 0.98 | 0 | 0.03413018 | 0.03538005 | 0.00341471 | −0.02946560 | −0.03071547 | 0 |
| 1.00 | 0 | 0.03252669 | 0.03252669 | 0 | −0.03252669 | −0.03252669 | 0 |

**Table 6**
$U$ (Downwelling at BOA) for $\tau = 0.5$, $\mu_0 = 0.2$, and $A = 0.0$

| $\mu$ | $\varphi = 0°$ | $\varphi = 30°$ | $\varphi = 60°$ | $\varphi = 90°$ | $\varphi = 120°$ | $\varphi = 150°$ | $\varphi = 180°$ |
|---|---|---|---|---|---|---|---|
| 0.02 | 0 | 0.00677002 | 0.01213073 | 0.01464574 | 0.01323643 | 0.00787572 | 0 |
| 0.06 | 0 | 0.00679905 | 0.01322260 | 0.01754947 | 0.01717397 | 0.01075042 | 0 |
| 0.10 | 0 | 0.00646972 | 0.01407278 | 0.02077195 | 0.02190528 | 0.01430222 | 0 |
| 0.16 | 0 | 0.00465948 | 0.01344137 | 0.02399258 | 0.02811500 | 0.01933310 | 0 |
| 0.20 | 0 | 0.00282059 | 0.01189742 | 0.02479837 | 0.03105461 | 0.02197778 | 0 |
| 0.28 | 0 | −0.00132035 | 0.00760602 | 0.02438731 | 0.03463403 | 0.02570766 | 0 |
| 0.32 | 0 | −0.00336870 | 0.00527877 | 0.02362532 | 0.03564148 | 0.02699402 | 0 |
| 0.40 | 0 | −0.00717886 | 0.00073690 | 0.02162626 | 0.03672088 | 0.02880512 | 0 |
| 0.52 | 0 | −0.01208088 | −0.00540771 | 0.01823144 | 0.03698549 | 0.03031232 | 0 |
| 0.64 | 0 | −0.01616543 | −0.01074985 | 0.01479565 | 0.03637667 | 0.03096109 | 0 |
| 0.72 | 0 | −0.01856617 | −0.01398952 | 0.01250364 | 0.03564645 | 0.03106981 | 0 |
| 0.84 | 0 | −0.02190697 | −0.01865230 | 0.00889191 | 0.03405355 | 0.03079888 | 0 |
| 0.92 | 0 | −0.02417402 | −0.02195936 | 0.00605055 | 0.03243922 | 0.03022456 | 0 |
| 0.96 | 0 | −0.02547988 | −0.02394259 | 0.00419995 | 0.03121711 | 0.02967982 | 0 |
| 0.98 | 0 | −0.02628920 | −0.02521200 | 0.00294297 | 0.03030938 | 0.02923217 | 0 |
| 1.00 | 0 | −0.02793563 | −0.02793563 | 0 | 0.02793563 | 0.02793563 | 0 |

**Table 7**
$X_l$, $Y_l$, $X_r$, and $Y_r$ for $\tau = 0.5$ and $\mu_0 = 0.2$

| Source | $X_l$ | $Y_l$ | $X_r$ | $Y_r$ |
|---|---|---|---|---|
| This work | 1.58755189 | 0.08404461 | 1.13216316 | 0.14626667 |
| Chandrasekhar & Elbert 1954 | 1.58729 | 0.08403 | 1.13214 | 0.14624 |

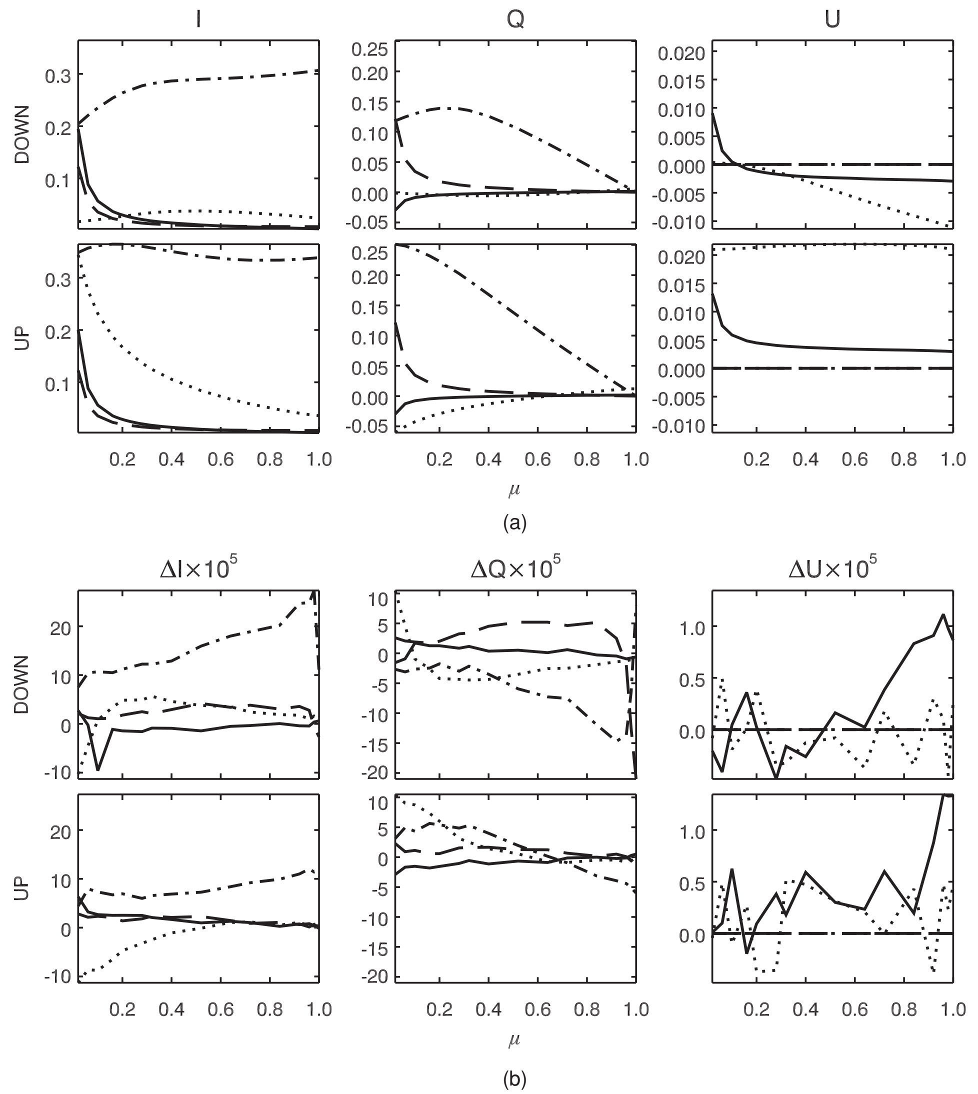



**Figure 1.** (a) Stokes parameters as functions of $\mu$ for $\varphi = 30°$ and $A = 0.0$. $(\tau, \mu_0) = (0.02, 0.1)$ (solid), $(0.02, 1.0)$ (dashed), $(1.0, 0.1)$ (dotted), and $(1.0, 1.0)$ (dash-dotted); (b) difference (multiplied by $10^5$) of Coulson et al. (1960) results from the present work.

**Table 8**
$X^1$, $Y^1$, $X^2$, and $Y^2$ for $\tau = 0.5$ and $\mu_0 = 0.2$

| Source | $X^1$ | $Y^1$ | $X^2$ | $Y^2$ |
|---|---|---|---|---|
| This work | 1.17743246 | 0.18156216 | 1.12706734 | 0.163347199 |
| Chandrasekhar & Elbert 1954 | 1.17742 | 0.18154 | 1.12705 | 0.16333 |

The initial conditions are the following:

$$X(\mu) = Y(\mu) = 1 \quad (\tau = 0, \mu > 0). \tag{21}$$

Approximating the integrals by $N$-point Gaussian quadrature, we obtain

$$\frac{dX(\mu_i)}{d\tau} = Y(\mu_i) \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{Y(\mu_j)}{\mu_j} \quad (i = 1, \ldots, N) \tag{22a}$$

$$\frac{dY(\mu_i)}{d\tau} = -\frac{Y(\mu_i)}{\mu_i} + X(\mu_i) \times \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{Y(\mu_j)}{\mu_j} \; (i = 1, \ldots, N), \tag{22b}$$

with initial conditions

$$X(\mu_i) = Y(\mu_i) = 1 \; (\tau = 0) \qquad (i = 1, \ldots, N). \tag{23}$$

$\mu_j$ and $w_j$ ($j = 1, \ldots, N$) are, respectively, the Gaussian quadrature points and weights.

Equations (22) can be integrated numerically using a variety of techniques (including, but not limited to, Runge–Kutta and Adams–Moulton algorithms). There are a few advantages of this technique over using the integral equations. First, convergence and nonuniqueness issues associated with the use of the integral equations (Chandrasekhar 1960; Carlstedt & Mullikin 1966) are not an issue here. Second, the limits of the $X$ and $Y$ functions for semi-infinite media are known, and can be used as tests of the precision of the technique.

$$\lim_{\tau \to \infty} X(\mu_i) = H(\mu_i) \qquad (i = 1, \ldots, N) \tag{24a}$$

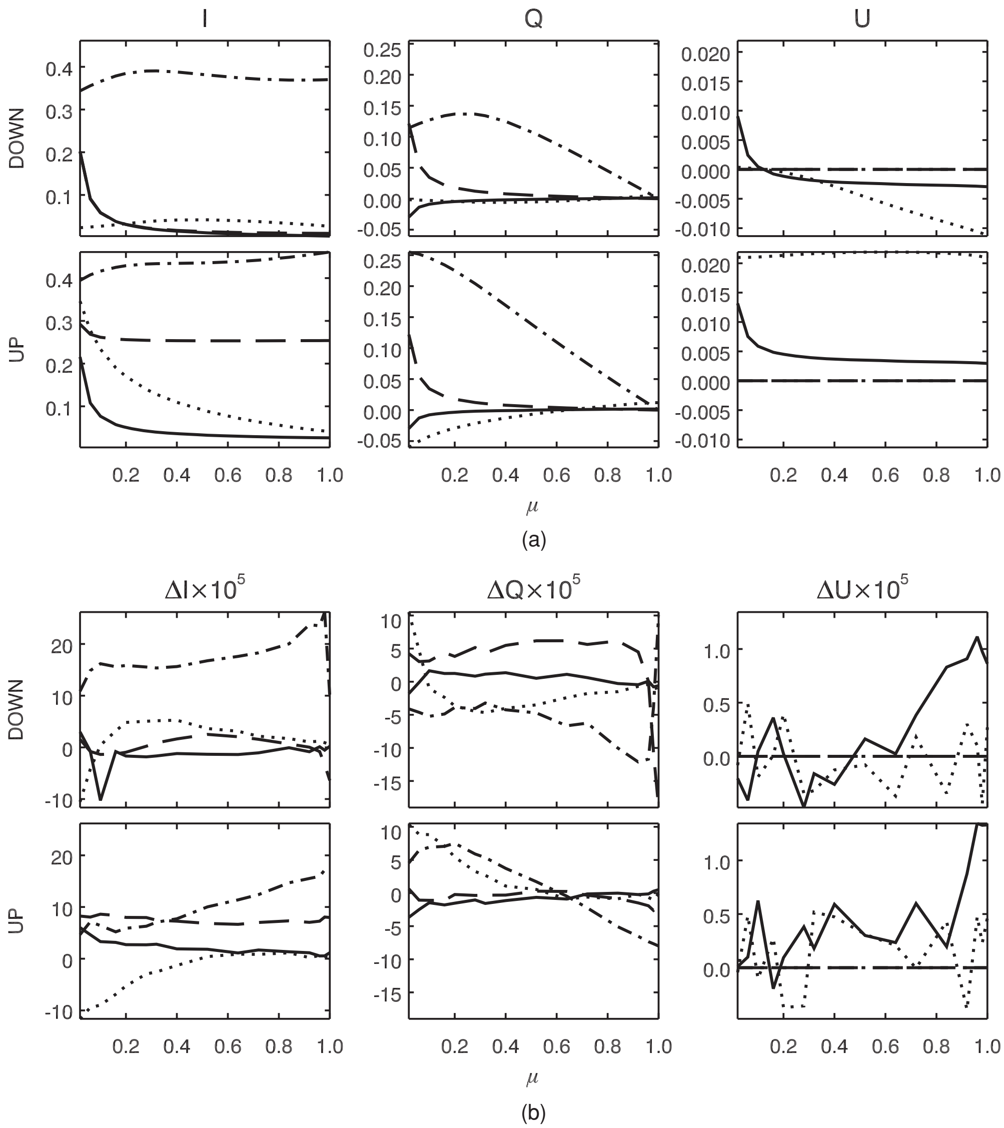



**Figure 2.** Same as Figure 1 except $A = 0.25$.

$$\lim_{\tau\to\infty} Y(\mu_i) = 0 \qquad (i = 1, \ldots, N). \tag{24b}$$

The right-hand side of Equation (24a) refers to the Chandrasekhar $H$ functions (Chandrasekhar 1960).

We used the fourth-order Adams–Moulton predictor–corrector algorithm (Moreno-Eguilaz et al. 1994) to solve Equations (22) to obtain the $X$ and $Y$ functions at the Gaussian quadrature points. However, these functions need to be evaluated at other angles. For this, we introduce the functions

$$\xi^{\pm}(\mu) = \int_0^1 \Psi(\mu') \frac{X(\mu')}{\mu \pm \mu'} d\mu' \tag{25a}$$

$$\zeta^{\pm}(\mu) = \int_0^1 \Psi(\mu') \frac{Y(\mu')}{\mu \pm \mu'} d\mu'. \tag{25b}$$

Note that the Cauchy principal values need to be used for $\xi^-$ and $\zeta^-$.

$\xi^+$ and $\zeta^+$ can be calculated using Gaussian quadrature

$$\xi^+(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{X(\mu_j)}{\mu + \mu_j} \tag{26a}$$

$$\zeta^+(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{Y(\mu_j)}{\mu + \mu_j}. \tag{26b}$$

However, Gaussian quadrature is not well suited to treating the singularity in the integrals for $\xi^-$ and $\zeta^-$ since the value of the integral is defined by symmetrical approach to the singular point. Here, we use the technique of addition and subtraction (Fuller & Hyett 1968)

$$\xi^-(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{X(\mu_j) - X(\mu)}{\mu - \mu_j} + X(\mu) \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} d\mu' \tag{27a}$$

$$\zeta^-(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{Y(\mu_j) - Y(\mu)}{\mu - \mu_j} + Y(\mu) \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} d\mu. \tag{27b}$$

The integral involving only the characteristic function can be analytically evaluated. The above expressions contain the unknown quantities $X(\mu)$ and $Y(\mu)$. However, when Equations (26) and (27) are substituted in Equations (9), we

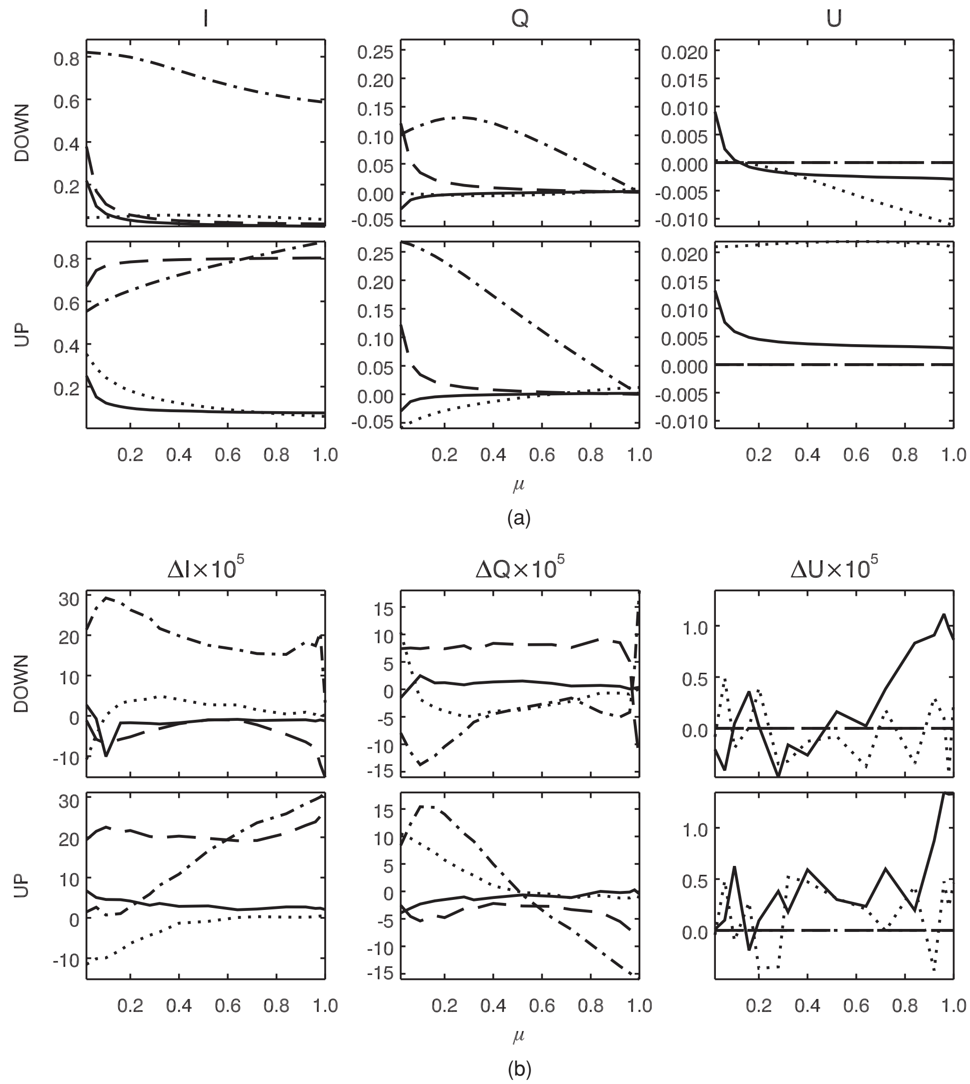



**Figure 3.** Same as Figure 1 except $A = 0.8$.

obtain

$$[1 - \mu\xi^{+}(\mu)]X(\mu) + \mu\zeta^{+}(\mu)Y(\mu) = 1 \quad (28a)$$

$$\mu\zeta_1^{-}(\mu)X(\mu) + [1 - \mu\xi_1^{-}(\mu)]Y(\mu) = e^{-\tau/\mu}, \quad (28b)$$

where $\xi_1^-$ and $\zeta_1^-$ are given by

$$\xi_1^{-}(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{X(\mu_j)}{\mu - \mu_j} \quad (29a)$$

$$\zeta_1^{-}(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{Y(\mu_j)}{\mu - \mu_j}. \quad (29b)$$

Equations (28) can be solved to obtain the required quantities. The above analysis is sufficient to solve the Rayleigh scattering problem provided $\mu \neq \mu_0$. In the special case of $\mu = \mu_0$, there is a singularity in the calculation of the transmission functions. Here, we appeal to L'Hôpital's rule. The solution now involves derivatives of the $X$ and $Y$ functions. Differentiating Equations (9), we obtain

$$[1 - \mu\xi^{+}(\mu)]X'(\mu) + \mu\zeta^{+}(\mu)Y'(\mu) = \xi^{+}(\mu)X(\mu) - \zeta^{+}(\mu)Y(\mu) - \mu\varsigma^{+}(\mu)X(\mu) + \mu\eta^{+}(\mu)Y(\mu) \quad (30a)$$

$$\mu\zeta^{-}(\mu)X'(\mu) + [1 - \mu\xi^{-}(\mu)]Y'(\mu) = \frac{\tau}{\mu^2}e^{-\tau/\mu} - \zeta^{-}(\mu)X(\mu) + \xi^{-}(\mu)Y(\mu) + \mu\eta^{-}(\mu)X(\mu) - \mu\varsigma^{-}(\mu)Y(\mu), \quad (30b)$$

where $\varsigma^{\pm}$ and $\eta^{\pm}$ are defined as follows:

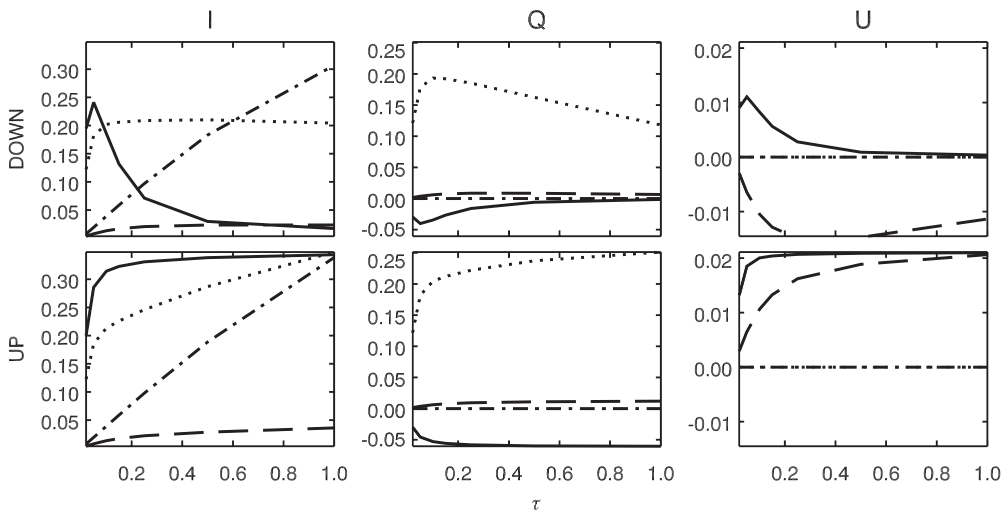
$$\varsigma^{\pm}(\mu) = \int_0^1 \Psi(\mu') \frac{X(\mu')}{(\mu \pm \mu')^2} d\mu' \quad (31a)$$

$$\eta^{\pm}(\mu) = \int_0^1 \Psi(\mu') \frac{Y(\mu')}{(\mu \pm \mu')^2} d\mu'. \quad (31b)$$

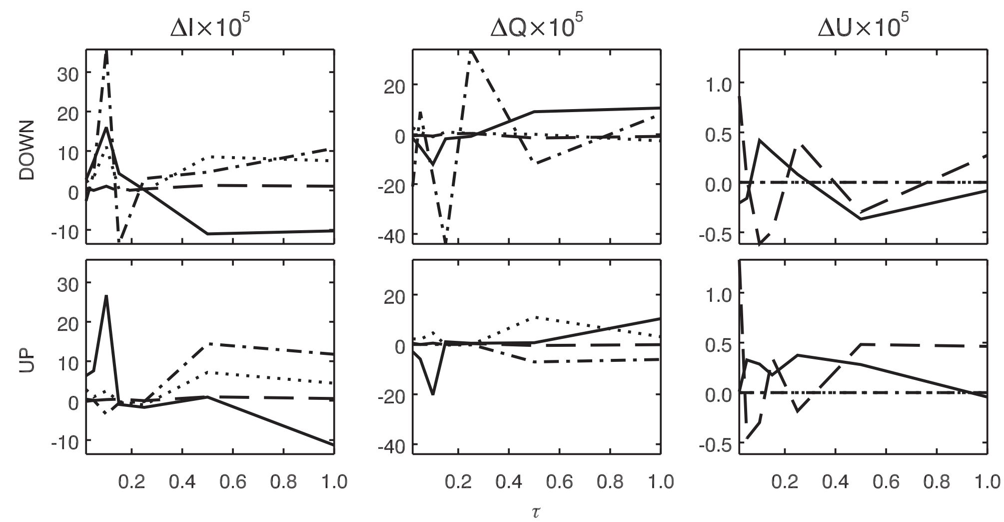
$\varsigma^+$ and $\eta^+$ can be calculated using Gaussian quadrature

$$\varsigma^{+}(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{X(\mu_j)}{(\mu + \mu_j)^2} \quad (32a)$$

$$\eta^{+}(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{Y(\mu_j)}{(\mu + \mu_j)^2}. \quad (32b)$$




(a)




(b)

**Figure 4.** (a) Stokes parameters as functions of $\tau$ for $\varphi = 30^\circ$ and $A = 0.0$. $(\mu_0, \mu) = (0.1, 0.02)$ (solid), $(0.1, 1.0)$ (dashed), $(1.0, 0.02)$ (dotted), and $(1.0, 1.0)$ (dash-dotted); (b) difference (multiplied by $10^5$) of Coulson et al. (1960) results from the present work.

The integrals for $\varsigma^-$ and $\eta^-$ are not defined even in the principal value sense. Ioakimidis (1981) showed that derivatives of Cauchy principal value integrals can be evaluated by formal differentiation of the quadrature rule for the corresponding principal value integral. $\varsigma^-$ and $\eta^-$ can thus be evaluated by term-by-term differentiation of the right-hand side of Equations (27)

$$\varsigma^-(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \left[ \frac{X(\mu_j) - X(\mu)}{(\mu - \mu_j)^2} + \frac{X'(\mu)}{\mu - \mu_j} \right] + X(\mu) \int_0^1 \frac{\Psi(\mu')}{(\mu - \mu')^2} d\mu' - X'(\mu) \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} d\mu' \quad (33a)$$

$$\eta^-(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \left[ \frac{Y(\mu_j) - Y(\mu)}{(\mu - \mu_j)^2} + \frac{Y'(\mu)}{\mu - \mu_j} \right] + Y(\mu) \int_0^1 \frac{\Psi(\mu')}{(\mu - \mu')^2} d\mu - Y'(\mu) \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} d\mu'. \quad (33b)$$

Substituting Equations (33) into Equation (30b), we obtain, after simplification

$$\mu \zeta_1^-(\mu) X'(\mu) + [1 - \mu \xi_1^-(\mu)] Y'(\mu) = \frac{\tau}{\mu^2} e^{-\tau/\mu} - \zeta_1^-(\mu) X(\mu) + \xi_1^-(\mu) Y(\mu) + \mu \eta_1^-(\mu) X(\mu) - \mu \varsigma_1^-(\mu) Y(\mu), \quad (34)$$

where $\varsigma_1^-$ and $\eta_1^-$ are given by

$$\varsigma_1^-(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{X(\mu_j)}{(\mu - \mu_j)^2} \quad (35a)$$

$$\eta_1^-(\mu) = \sum_{j=1}^{N} w_j \Psi(\mu_j) \frac{Y(\mu_j)}{(\mu - \mu_j)^2}. \quad (35b)$$

Equations (34) and (30a) can be solved to obtain the derivatives of the $X$ and $Y$ functions. A final point to note is that the characteristic function $\Psi_l$ is conservative (Chandrasekhar 1960)

$$\int_0^1 \Psi_l(\mu)\, d\mu = \frac{1}{2}. \quad (36)$$

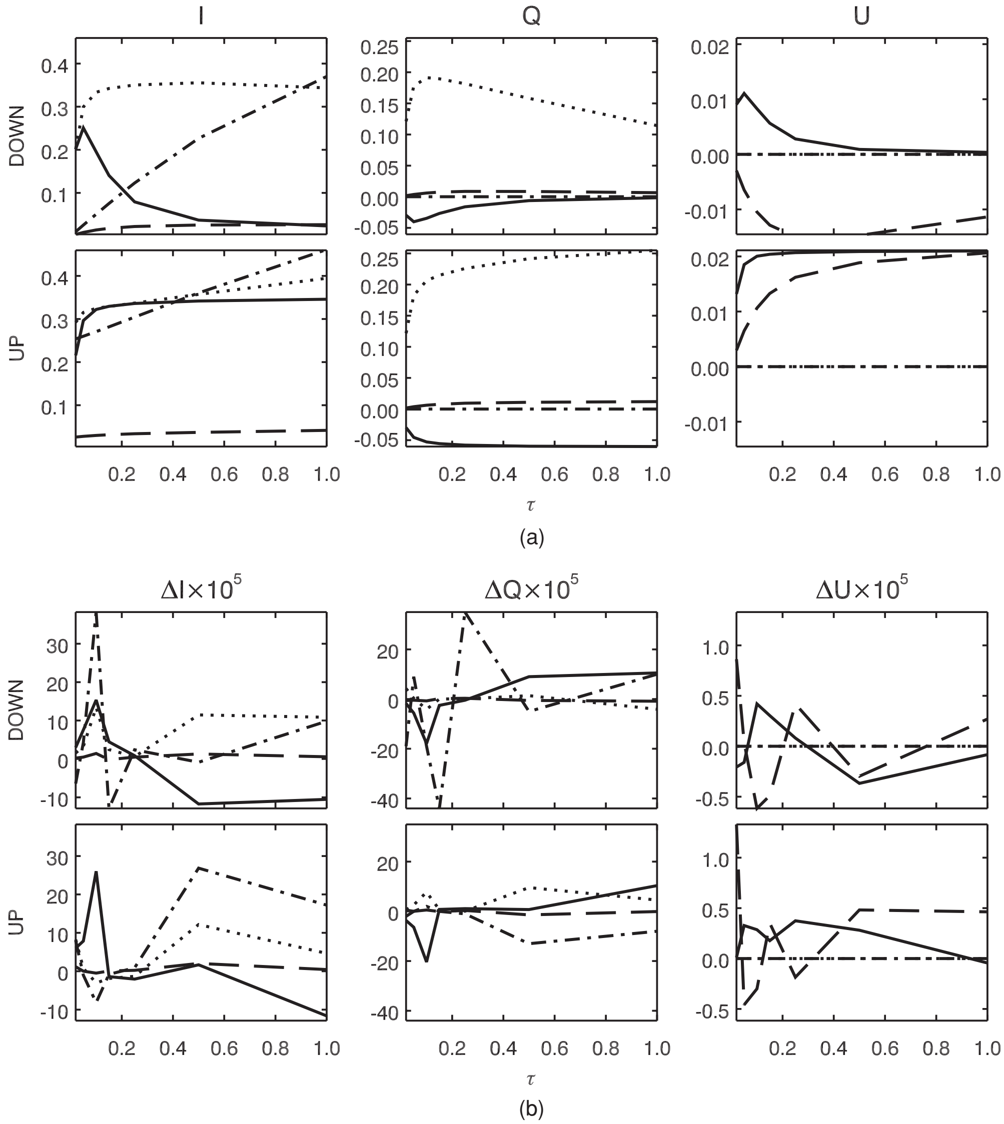



**Figure 5.** Same as Figure 4 except $A = 0.25$.

In this case, the $X$ and $Y$ functions refer to the standard solution, having the property

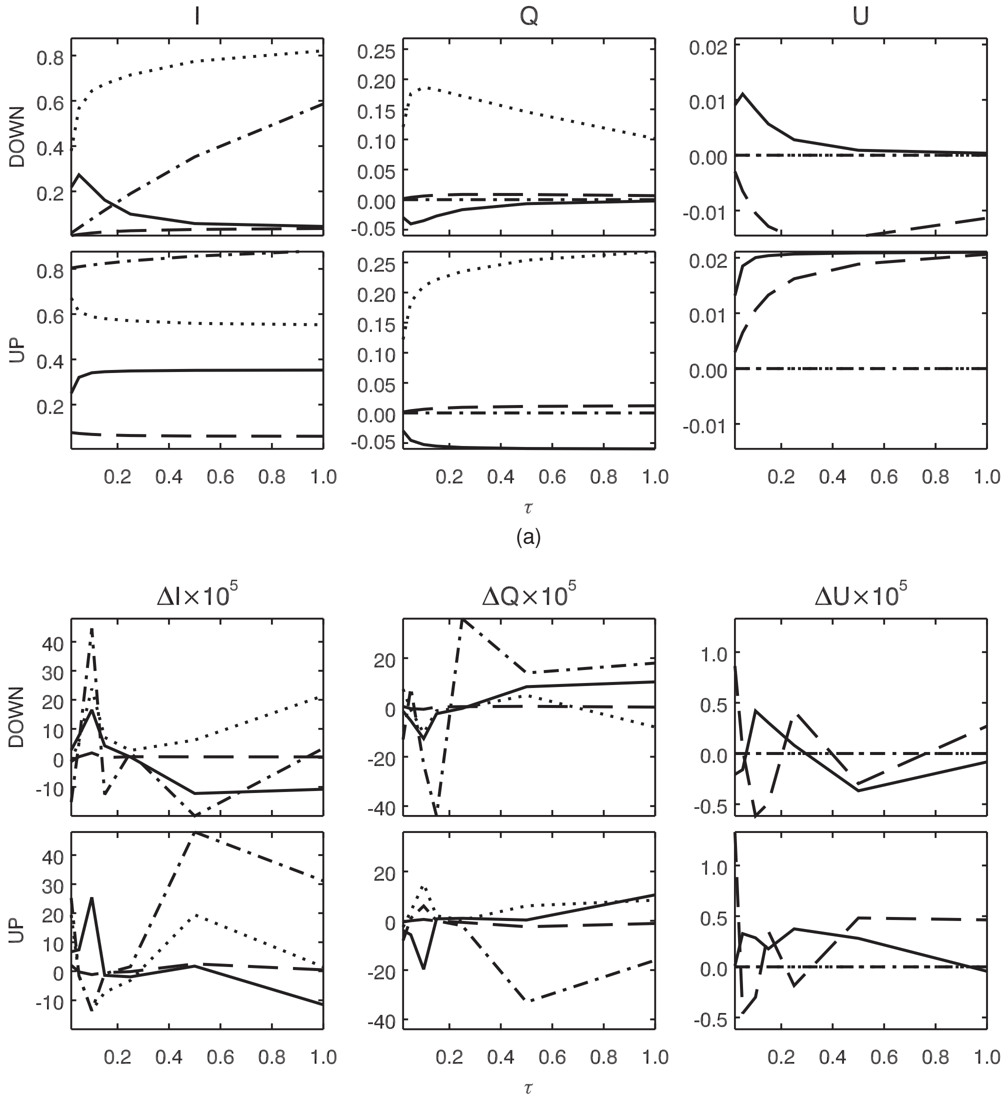
$$\int_0^1 \Psi_l(\mu) X_l(\mu)\, d\mu = 1 \tag{37a}$$

$$\int_0^1 \Psi_l(\mu) Y_l(\mu)\, d\mu = 0. \tag{37b}$$

The standard solution can be obtained from any particular solution $\left(X_l^p, Y_l^p\right)$ by solving for a constant $z$, such that $\left(X_l^p + z\mu\left[X_l^p + Y_l^p\right], Y_l^p - z\mu\left[X_l^p + Y_l^p\right]\right)$ satisfy Equations (37).

## 4. RESULTS AND DISCUSSION

We tabulate results for $I$ (Tables 1 and 2), $Q$ (Tables 3 and 4), and $U$ (Tables 5 and 6) for benchmarking purposes. In Tables 1–6, $\tau = 0.5$, $\mu_0 = 0.2$, and $A = 0.0$. The complete tables of all the relevant Stokes parameters can be obtained from the Web site: http://www.gps.caltech.edu/~vijay/Rayleigh_Scattering_Tables. In our calculations, we assumed the flux parameter $F_0$ to be unity. To understand the differences between our results and those of Coulson et al. (1960), we plot the Stokes parameters as functions of $\mu$ (Figures 1, 2, and 3) and $\tau$ (Figures 4, 5, and 6). In each of these figures, the values of the parameters (as per our computations) are first plotted, followed by the residuals between the Coulson et al. (1960) values and our values. The residuals have been multiplied by $10^5$ to show the difference in the fifth decimal place. For example, a residual of 5 implies that the Coulson et al. (1960) tables are wrong by five units in the fifth decimal place. Two things are evident. First, there is no real pattern to the errors. Second, the errors are much more for Stokes parameters $I$ and $Q$ than for $U$. This indicates that the largest error is in the computation of the zeroth Fourier component (which does not contribute to $U$). To confirm this, we compute the various $X$ and $Y$ functions and compare them with the results obtained by Chandrasekhar & Elbert (1954), which were used by Coulson et al. (1960) in their analysis. Tables 7 and 8 validate our hypothesis—the largest




(a)

(b)

**Figure 6.** Same as Figure 4 except $A = 0.8$.

error is indeed for $X_l$, which is involved in the computation of the zeroth Fourier moment. It is interesting to note that the characteristic function for $X_l$ is conservative.

All the results have been tabulated to eight decimal places. We confirmed the precision of our results by performing the following tests: (1) increasing the number of quadrature points; (2) increasing the number of integration subintervals; (3) using a different numerical integration technique; (4) comparing with results from the multiple scattering radiative transfer code VLIDORT (Spurr 2006); and (5) computing the $X$ and $Y$ functions for very large optical depths and comparing the results with Equations (24).

It is clear that the Rayleigh scattering tables of Coulson et al. (1960) are not accurate even to the degree indicated in their work (one or two units in the fifth decimal place). For example, their results for upwelling intensity at TOA were off by three units in the fourth decimal place for the following parameter values: $\tau = 0.5$, $\mu_0 = 1.0$, $\mu = 1.0$, $\varphi = 0°$, and $A = 0.8$. There are several causes for this. First, they obtained several quantities from tabulations of limited precision. For example, Chandrasekhar & Elbert (1954) computed most quantities only to five decimal places. Second, the integral equations used to solve for the $X$ and $Y$ functions had inherent convergence issues. Third, the derivatives of the $X$ and $Y$ functions required for computing the transmission functions in the solar alucantar were obtained numerically. Finally, the calculations were probably performed using single-precision arithmetic.

## 5. CONCLUSIONS

The Rayleigh scattering radiative transfer problem was solved starting from the integro-differential form of the $X$ and $Y$ functions. A fourth-order Adams–Moulton predictor–corrector method was employed to solve for the $X$ and $Y$ functions. Singular integrals arising during the computation of these functions at user-specified angles were evaluated using standard techniques for Cauchy principal value integrals. Further, derivatives of these integrals were evaluated by formal differentiation of the quadrature rule for the corresponding Cauchy principal value integrals. This technique was shown to give results accurate to eight decimal places, even for direct transmission in the direction of the incoming radiation.

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