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A Note on the Variational Method of Stokes and DeMarcus for Radiative Transfer in Planetary Atmospheres[1]

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A generalized functional which yields the Milne integral equation on variation and whose extremum value is proportional to the reflectivity at arbitrary emergent angle is proposed. A similar functional exists for computing the transmissivity at arbitrary emergent angle. This work is a generalization of the variational method of Stokes and DeMarcus (1971, *Icarus* **14,** 307) based on the principle of reciprocity. In the special case of trial functions that are linear in the undetermined parameters, the calculation is greatly simplified. The computational value of our variational principle is demonstrated.

INTRODUCTION

A number of authors (Huang, 1952a,b; Stokes and DeMarcus, 1971; Sze, 1976) have employed the variational method for solving the planetary problem in radiative transfer. Huang worked on conservative scattering in a homogeneous isotropic atmosphere of infinite depth. He constructed a functional which, upon variation, yields the integral equation first derived by Milne. A linear combination of five simple functions was used as the trial source function; and an accuracy of around a few parts per million was achieved for the specific intensity. Stokes and DeMarcus (1971) first pointed out that the reflectivity in the backward direction is proportional to the extremum value of a functional. The reflectivity is then accurate to second order. For isotropic scattering in a homogeneous atmosphere, they used a second-order polynomial trial source function and achieved an accuracy of a few parts in ten thousand for the reflectivity in the backward direction. Sze (1976) used a hybrid of variational and iterative methods to calculate the reflectivity of a finite atmosphere at arbitrary emergent angles. Without the full benefit of a functional, his trial source functions have to be much more sophisticated in order to match Stokes and DeMarcus' accuracy.

In this paper we propose a generalization of the variational principle of Stokes and DeMarcus (1971) by the construction of two bifunctionals. The equation of radiative transfer is obtained by imposing the stationary condition on such functionals and the reflectivity and transmissivity at arbitrary angle are given by their extremum values. Further simplification is possible when one uses trial functions that are linear in the variational parameters, in which case the source function derived from the method of Stokes and DeMarcus can give the reflectivity at arbitrary emergent angle to second order. Finally, we demonstrate the usefulness and explore the limitations of our variational principles with extensive numerical results.

THEORY

Following the notation of Chandrasekhar (1960), the equation of transfer in a plane-parallel atmosphere with an isotropic scatterer is

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$$\mu \frac{\partial I}{\partial \tau}(\tau,\mu) = I(\tau,\mu) - \omega_0(\tau)J(\tau), \quad (1)$$

where

$$J(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\, d\mu$$

$$= \frac{1}{2}\int_{0}^{1} [I^+(\tau,\mu) + I^-(\tau,\mu)]\, d\mu$$

$I^+(\tau,\mu) = I(\tau,\mu) \qquad 0 \le \mu \le 1$

$I^-(\tau,\mu) = I(\tau,-\mu) \qquad 0 \le \mu \le 1$

τ = normal optical depth

μ = cosine of zenith angle

$\omega_0(\tau)$ = single scattering albedo.

$J(\tau)$ is the mean intensity. It is directly proportional to the source function for an isotropically scattering atmosphere. $J(\tau)$ satisfies the Milne integral equation

$$J(\tau) = \frac{F}{4\mu_0} e^{-\tau/\mu_0} + \frac{1}{2}\int_0^{\tau_1} \omega_0(s)J(s)E_1|\tau - s|\, ds \quad (2)$$

where

μ_0 = cosine of the solar zenith angle

πF = incident solar flux normal to the atmosphere (note that πF in Chandrasekhar (1960) is defined to be normal to the direction of propagation)

$$E_n(x) = \int_0^1 e^{-x/\mu}\mu^{n-2}\, d\mu.$$

The solution of Eq. (1) is related to $J(\tau)$ by

$$I^+(\tau,\mu) = \int_\tau^{\tau_1} \omega_0(s)J(s)e^{-(s-\tau)/\mu}\frac{ds}{\mu} \quad (3a)$$

$$I^-(\tau,\mu) = \frac{F}{2\mu_0} e^{-\tau/\mu_0}\,\delta(\mu - \mu_0) + \int_0^{\tau} \omega_0(s)J(s)e^{-(\tau-s)/\mu}\frac{ds}{\mu}. \quad (3b)$$

In particular we are interested in calculating the reflectivity $R(\mu,\mu_0) = I^+(\tau = 0,\mu,\mu_0)/F$ and the transmissivity $T(\mu,\mu_0) = I^-(\tau_1,\mu,\mu_0)/F$ (where the first angle is the emergent zenith angle and the second is the solar zenith angle).

Now let $\alpha(t)$ and $\beta(t)$ be two arbitrary integrable functions. Consider the functional,

$$W(\alpha,\beta) = \int_0^{\tau_1} [\alpha(s)S(s,\mu) + \beta(s)S(s,\mu_0)]\, ds + \int_0^{\tau_1}\int_0^{\tau_1} \alpha(t)G(t,s)\beta(s)\, dsdt - \int_0^{\tau_1} \alpha(s)\beta(s)\, ds \quad (4)$$

where

$$S(s,\mu) = \frac{F}{4\mu} e^{-s/\mu}\sqrt{\omega_0(s)}$$

$$G(t,s) = \frac{1}{2}\sqrt{\omega_0(s)\omega_0(t)}\, E_1|t - s|. \quad (5)$$

This functional is a generalization of the one written by Stokes and DeMarcus (1971). Imposing the condition that W be stationary with respect to variation in $\alpha(t)$ and $\beta(t)$ yields two equations,

$$\alpha(\tau) = S(\tau,\mu_0) + \int_0^{\tau_1} G(\tau,s)\alpha(s)\, ds \quad (6a)$$

and

$$\beta(\tau) = S(\tau,\mu) + \int_0^{\tau_1} G(\tau,s)\beta(s)\, ds, \quad (6b)$$

where the functions S and G are as given in Eq. (5). α and β can be identified as proportional to the mean intensity for sunlight incident at angles μ_0 and μ, respectively. In fact, upon putting

$$\alpha(\tau) = \sqrt{\omega_0(\tau)}J(\tau)$$

$$\beta(\tau) = \sqrt{\omega_0(\tau)}K(\tau)$$

and substituting the expressions for α, β, S, and G into (6a) and (6b) we get

$$J(\tau) = \frac{F}{4\mu_0} e^{-\tau/\mu_0} + \frac{1}{2}\int_0^{\tau_1} \omega_0(s)J(s)E_1|\tau - s|\, ds \quad (6c)$$

and

$$K(\tau) = \frac{F}{4\mu} e^{-\tau/\mu} + \frac{1}{2}\int_0^{\tau_1} \omega_0(s) K(s) E_1|\tau - s|\, ds. \quad (6d)$$

$J(\tau)$ is then readily identified as the mean intensity when sunlight is incident on the atmosphere at an angle μ_0 and $K(\tau)$ is the corresponding mean intensity when the incident angle is μ. πF is the flux normal to the atmosphere. See Fig. 1 for details.

The extremum value of W is obtained by substituting (6c) into (4); we have

$$W_{ex} = \int_0^{\tau_1} \frac{F}{4\mu} e^{-s/\mu} \omega_0(s) J(s)\, ds = \frac{F^2}{4} R(\mu,\mu_0). \quad (7)$$

The principle of reciprocity (Minnaert, 1941; van de Hulst, 1980) requires

$$R(\mu,\mu_0) = R(\mu_0,\mu) = \frac{4}{F^2}\int_0^{\tau_1} \frac{F}{4\mu_0} e^{-s/\mu_0} \omega_0(s) K(s)\, ds \quad (8)$$

provided that the incident normal flux component is the same. Reciprocity has been incorporated into the functional by virtue of its symmetry in μ and μ_0. Equation (8) can be derived from the functional by substituting (6d) into (4) and comparing it with (7). On setting $\mu = \mu_0$ in $W(\alpha,\beta)$, our results reduce to those of Stokes and DeMarcus (1971). In principle, the mean intensity derived from their functional can also be used to calculate the reflectivity at any zenith angle using Eq. (3a). The advantage of the functional $W(\alpha,\beta)$ is that its extremum value gives the reflectivity to second order in the error of the trial α and β. We surmise that as a direct consequence of the principle of reciprocity, similar variational methods exist for transfer problems with arbitrary phase functions. Unfortunately, we are at present unable to ascertain the validity of our conjecture in this work.

A similar functional can be written which

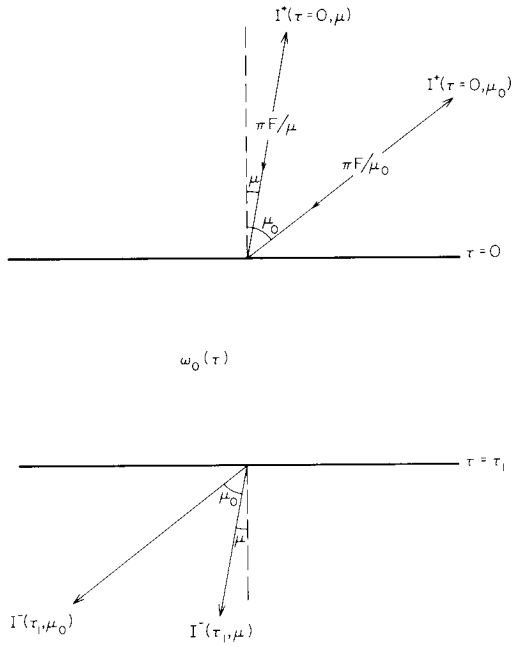

FIG. 1. Schematic diagram showing the geometry of the radiative transfer problem. The incident solar fluxes given are normal to the direction of propagation (see text).

yields the transmissivity $T(\mu,\mu_0)$ on variation,

$$W'(J,K) = \int_0^{\tau_1} \omega_0(s) K(s) \frac{F}{4\mu_0} e^{-s/\mu_0}\, ds + \int_0^{\tau_1} \omega_0(s) J(s) \frac{F}{4\mu} e^{-(\tau_1 - s)/\mu}\, ds + \int_0^{\tau_1}\int_0^{\tau_1} J(t) K(s) \frac{1}{2} \omega_0(s)\omega_0(t) E_1|t - s|\, ds\, dt - \int_0^{\tau_1} \omega_0(s) J(s) K(s)\, ds.$$

Since we are solving the same problem, this functional must give the same $J(t)$ and $K(t)$. A little reflection reveals that the same reasoning leading to (7) and (8) also applies here, giving us

$$T(\mu,\mu_0) = T(\mu_0,\mu) = \frac{4}{F^2} W'_{ex}.$$

ANALYSIS OF ACCURACY

Analytic expressions for $J(\tau)$ and $K(\tau)$ can be obtained by choosing some trial functions

$$J_t(\tau) = J_t(a_1, a_2, \ldots, a_n, \tau)$$
$$K_t(\tau) = K_t(b_1, b_2, \ldots, b_n, \tau)$$

with $\{a_i\} = a_1, \ldots, a_n$ and $\{b_i\} = b_1, \ldots, b_n$ as unknown parameters to be determined by imposing the extremum condition on W,

$$\frac{\partial W}{\partial a_i} = \frac{\partial W}{\partial b_j} = 0. \tag{9}$$

With $J_t(\tau)$ and $K_t(\tau)$ determined, $R(\mu,\mu_0)$ can be calculated by the three ways:

$$R_1(\mu,\mu_0) = \frac{1}{F}\int_0^{\tau_1} K_t(s)\omega_0(s)e^{-s/\mu_0}\frac{ds}{\mu_0} \tag{10a}$$

$$R_2(\mu,\mu_0) = \frac{1}{F}\int_0^{\tau_1} J_t(s)\omega_0(s)e^{-s/\mu}\frac{ds}{\mu} \tag{10b}$$

$$R_3(\mu,\mu_0) = \frac{4}{F^2}W(\sqrt{\omega_0}J_t, \sqrt{\omega_0}K_t). \tag{10c}$$

Intuitively, one expects R_3 to be the most accurate since R_1 and R_2 are accurate to first order, while R_3 is accurate to second order as shown in

$$\delta R_1 = \frac{1}{F}\int_0^{\tau_1} \delta K(s)\omega_0(s)e^{-s/\mu}\frac{ds}{\mu_0} \tag{11a}$$

$$\delta R_2 = \frac{1}{F}\int_0^{\tau_1} \delta J(s)\omega_0(s)e^{-s/\mu}\frac{ds}{\mu} \tag{11b}$$

$$\delta R_3 = -\frac{4}{F^2}\int_0^{\tau_1} \delta J(s)\delta K(s)\omega_0(s)\,ds + \frac{4}{F^2}\int_0^{\tau_1}\int_0^{\tau_1}\sqrt{\omega_0(t)}\delta J(t)G(t,s)\sqrt{\omega_0(s)}\delta K(s)\,ds\,dt \tag{11c}$$

where

$$\delta J(s) = J_t(s) - J_e(s)$$
$$J_t = \text{trial } J$$
$$J_e = \text{exact } J.$$

Note that δR_3 is indeed of second order because $\int_0^{\tau_1}\int_0^{\tau_1}|G(t,s)|\,ds\,dt$ is bounded from above. The validity of these predictions remains to be established by computations.

In actual calculations, it is convenient to choose trial functions that are linear in the undetermined parameters:

$$J_t(\tau) = \sum_j a_j\eta_j(\tau)$$
$$K_t(\tau) = \sum_j b_j\eta_j(\tau) \tag{12}$$

where $\{\eta_i\}$ is some convenient set of functions and j runs from 1 to a small integer. In this case there exists further simplification. The variational method for obtaining the $\{a_i\}$ and $\{b_i\}$ then gives

$$\begin{aligned}\frac{F^2}{4}R_3(\mu,\mu_0) &= W(\sqrt{\omega_0}J_t, \sqrt{\omega_0}K_t)\\ &= \frac{F^2}{4}R_2(\mu,\mu_0) + \sum_j b_j\frac{\partial W}{\partial b_j}\\ &= \frac{F^2}{4}R_2(\mu,\mu_0)\\ &= \frac{F^2}{4}R_1(\mu,\mu_0) + \sum_j a_j\frac{\partial W}{\partial a_j}\\ &= \frac{F^2}{4}R_1(\mu,\mu_0)\end{aligned} \tag{13}$$

where (9) has been used. R_1, R_2, and R_3 are therefore all equal.

Further,

$$\frac{\partial W}{\partial a_i}(a_j,b_k) = \frac{\partial W}{\partial a_i}(b_k)$$
$$\frac{\partial W}{\partial b_i}(a_j,b_k) = \frac{\partial W}{\partial b_i}(a_j) \tag{14}$$

i.e., the set of n equations $\partial W/\partial a_i = 0$ do not involve the a_i's and the other $n(\partial W/\partial b_i = 0)$ do not involve the b_i's. This simplification arises from the linearity of W in J and K and the linearity of J (and K) in the a_i's (and the b_i's). If one has chosen some nonlinear trial functional form for J or K, such as the Pade approximation (Gragg, 1972), then none of the above simplification will occur. The decoupling of the two sets of equations enables us to solve only one of them for $\{a_i\}$ (or $\{b_i\}$) in order to calculate R_2 (or R_1). The

$R(\mu,\mu_0)$ obtained is then accurate to second order.

Let us define T_1, T_2, and T_3 in a manner similar to R_1, R_2, and R_3. The derivation of (14) can be carried through for T exactly as before and we have,

$$W'(J_t,K_t) = \frac{F^2}{4} T_3(\mu,\mu_0)$$
$$= \frac{F^2}{4} T_2(\mu,\mu_0)$$
$$= \frac{F^2}{4} T_1(\mu,\mu_0).$$

In particular, $T_2(\mu,\mu_0)$ can be calculated as in (10b) by substituting $e^{-s/\mu}$ with $e^{-(\tau_1-s)/\mu}$.

There is a fundamental limitation to the accuracy of reflectivity and transmissivity evaluated by variational methods. At grazing angle, it can be shown (Chandrasekhar, 1960) that

$$FR(\mu = 0,\mu_0) = I^+(\tau = 0,\mu = 0,\mu_0)$$
$$= \omega_0(0)J(0)$$

and

$$FT(\mu = 0,\mu_0) = I^-(\tau = \tau_1,\mu = 0,\mu_0)$$
$$= \omega_0(\tau_1)J(\tau_1)$$

where $J(0)$ and $J(\tau_1)$ are the mean intensities evaluated at the upper and lower boundary, respectively. Here the accuracy of reflectivity and transmissivity is obviously of first order. The reason that R and T can be computed to higher accuracy than the source function is that they can be expressed as integrals of the source function. As shown by Sze (1976), the enhanced accuracy for R is due to the cancellation of errors in the integration. At grazing angle, there is no cancellation as the integral reduces to the function at the boundary. Thus we expect the accuracy of R and T to divide into approximately two regions; a first-order region for emergent angles $0 \leq \mu \leq \mu_c$ and a second order region for emergent angles $\mu_c \leq \mu \leq 1$. Numerical computations show that μ_c, the critical angle for transition between the two regimes, is about 0.2–0.3.

RESULTS AND DISCUSSION

A large number of simple cases have been studied to illustrate the ideas and formulae developed above. Table I is a list of the cases and their methods of derivation.

TABLE I

LISTING OF THE PHYSICAL PARAMETERS OF CASES STUDIED, AND THE METHODS USED[a]

Table	Case	τ_1	ω_0	μ_0	Method
2	A	1	1	1	Variational, three-term polynomial trial function
2	B	1	0.9	1	Variational, three-term polynomial trial function
2	C	1	0.6	1	Variational, three-term polynomial trial function
2	D	1	0.2	1	Variational, three-term polynomial trial function
2	E	1	1	0.1	Variational, three-term polynomial trial function
2	F	1	0.9	0.1	Variational, three-term polynomial trial function
2	G	4	1	1	Variational, six-term polynomial trial function
3	H	4	0.9	0.5	Variational, six-term polynomial trial function
3	I	4	1	0.3	Variational, six-term polynomial trial function
4	J	∞	1	1	Variational, five-term exponential and exponential integral functions as trial function
5	K	1	1	1	Variational, with three-step functions as trial functions, interactive improvement, result fitted to five-term polynomial
5	L	1	1	0.1	Variational, with three-step functions as trial functions, interactive improvement, result fitted to five-term polynomial

[a] The results are summarized in Tables II–V.

TABLE II

REFLECTION (R) AND TRANSMISSION (T) FUNCTIONS FOR A HOMOGENEOUS ATMOSPHERE WITH $\tau_1 = 1$ AND VARIOUS VALUES OF ω_0 AND μ_0 (CASES A–F)

μ	0	0.1	0.3	0.5	0.7	0.9	1.0	$F^{+,-}$
				Case A $\tau_1 = 1$, $\omega_0 = 1$, $\mu_0 = 1$				
R	0.463 17	0.467 91	0.436 71	0.380 42	0.328 82	0.286 96	0.269 32	0.341 37
R_0	0.439 34	0.464 88	0.436 66	0.380 51	0.328 91	0.287 03	0.269 39	0.341 33
Δ	0.023 83	0.003 03	0.000 05	−0.000 09	−0.000 09	−0.000 07	−0.000 07	0.000 04
T	0.243 37	0.292 06	0.336 24	0.320 93	0.290 32	0.260 20	0.246 54	0.658 63
T_0	0.241 51	0.293 40	0.336 36	0.320 91	0.290 30	0.260 18	0.246 53	0.658 67
Δ	0.001 86	−0.001 34	−0.000 12	0.000 02	0.000 02	0.000 02	0.000 01	−0.000 04
				Case B $\tau_1 = 1$, $\omega_0 = 0.9$, $\mu_0 = 1$				
R	0.377 24	0.374 99	0.344 31	0.298 10	0.256 92	0.223 84	0.209 96	0.267 46
R_0	0.359 37	0.372 67	0.344 24	0.298 15	0.256 97	0.223 88	0.210 00	0.267 41
Δ	0.017 87	0.002 32	0.000 07	−0.000 05	−0.000 05	−0.000 04	−0.000 04	0.000 05
T	0.187 29	0.223 01	0.257 48	0.246 68	0.223 65	0.200 71	0.190 27	0.591 59
T_0	0.186 14	0.224 03	0.257 58	0.246 68	0.223 63	0.200 70	0.190 26	0.591 63
Δ	0.001 15	−0.001 02	−0.000 10	0.000 00	0.000 02	0.000 01	0.000 01	−0.000 04
				Case C $\tau_1 = 1$, $\omega_0 = 0.6$, $\mu_0 = 1$				
R	0.200 54	0.191 96	0.169 38	0.144 42	0.123 57	0.107 21	0.100 41	0.129 54
R_0	0.194 11	0.191 12	0.169 33	0.144 43	0.123 58	0.107 22	0.100 42	0.129 51
Δ	0.006 43	0.000 84	0.000 05	−0.000 01	−0.000 01	−0.000 01	−0.000 01	0.000 03
T	0.087 39	0.101 43	0.117 66	0.113 80	0.103 75	0.093 43	0.088 68	0.471 36
T_0	0.086 91	0.101 76	0.117 70	0.113 80	0.103 75	0.093 43	0.088 68	0.471 37
Δ	0.000 48	−0.000 33	−0.000 04	0.000 00	0.000 00	0.000 00	0.000 00	−0.000 01
				Case D $\tau_1 = 1$, $\omega_0 = 0.2$, $\mu_0 = 1$				
R	0.054 10	0.050 03	0.042 48	0.035 67	0.030 29	0.026 17	0.024 47	0.031 98
R_0	0.053 70	0.050 00	0.042 47	0.035 67	0.030 29	0.026 17	0.024 47	0.031 98
Δ	0.000 40	0.000 03	0.000 01	0.000 00	0.000 00	0.000 00	0.000 00	0.000 00
T	0.021 08	0.023 61	0.027 38	0.026 73	0.024 51	0.022 15	0.021 05	0.392 26
T_0	0.020 88	0.023 59	0.027 38	0.026 73	0.024 51	0.022 15	0.021 05	0.392 26
Δ	0.000 20	0.000 02	0.000 00	0.000 00	0.000 00	0.000 00	0.000 00	0.000 00
				Case E $\tau_1 = 1$, $\omega_0 = 1$, $\mu_0 = 0.1$				
R	2.132 60	1.629 58	1.054 87	0.776 19	0.614 20	0.508 25	0.467 91	0.695 36
R_0	2.986 22	1.779 64	1.058 44	0.771 82	0.610 02	0.504 86	0.464 88	0.697 66
Δ	−0.853 62	−0.150 06	−0.003 57	0.004 37	0.004 18	0.003 39	0.003 03	−0.002 30
T	0.435 83	0.272 07	0.279 32	0.316 92	0.316 99	0.301 68	0.292 06	0.304 64
T_0	0.138 98	0.177 54	0.273 71	0.318 63	0.318 98	0.303 24	0.293 40	0.302 34
Δ	0.296 85	0.094 53	0.005 61	−0.001 71	−0.001 99	−0.001 56	−0.001 34	0.002 30
				Case F $\tau_1 = 1$, $\omega_0 = 0.9$, $\mu_0 = 0.1$				
R	1.826 56	1.377 19	0.870 65	0.631 78	0.495 83	0.408 13	0.374 98	0.566 00
R_0	2.614 00	1.516 58	0.875 57	0.628 83	0.492 72	0.405 55	0.372 67	0.568 99
Δ	−0.787 44	−0.139 39	−0.004 92	0.002 95	0.003 11	0.002 58	0.002 31	−0.002 99
T	0.360 17	0.204 00	0.200 39	0.234 86	0.238 97	0.229 61	0.223 01	0.228 34
T_0	0.091 56	0.116 90	0.194 25	0.235 79	0.240 40	0.230 78	0.224 03	0.225 75
Δ	0.268 63	0.087 10	0.006 14	−0.000 93	−0.001 43	−0.001 17	−0.001 02	0.002 59

Note. The variation principle with three-term polynomial is used. The "exact" values R_0 and T_0 are taken from van de Hulst (1980). The symbol Δ denotes the difference between "exact" and approximate solutions.

All integrals are performed numerically by Gaussian quadrature.

The column $F^{+,-}$ in the tables serves as a further check on our calculations. F^+ (entered in row R) is defined as

$$F^+ = 2\omega_0 \int_0^{\tau_1} E_2(s)J(s)\,ds,$$

which is the upward flux. F^- (entered in row T) is defined as

$$F^- = 2\omega_0 \int_0^{\tau_1} E_2(\tau_1 - s)J(s)\,ds + Fe^{-\tau_1/\mu_0},$$

which is another expression for the downward flux. Their "exact" values are taken from van de Hulst (1980).

Table II shows the results of calculations using a second-order polynomial trial function for an atmosphere of optical thickness $\tau_1 = 1$. ω_0 was chosen to be constant for convenience. Typically the error stays small (a few parts per ten thousand) until μ goes below 0.3. In case E and F, the solar zenith angle μ_0 is small and the error is significant as expected.

Table III contains results for an atmosphere of larger optical depth, $\tau_1 = 4$. A fifth-order polynomial trial function was used to achieve comparable accuracy as in Table II. Again, the error increases for $\mu \leq 0.3$.

For an infinite atmosphere, we took the results of Huang (1952b) and reconstructed his calculations to eight significant figures in Table IV. We used double precision on an IBM 370/3032 and achieved higher numerical accuracy than did Huang. Huang's choice of trial function gave him an accuracy of a few parts in ten million. Note also that Δ grows as μ goes to zero. The fluctuating sign of Δ for $\mu \leq 0.3$ has little significance because the last significant fi-

TABLE III

REFLECTION (R) AND TRANSMISSION (T) FUNCTIONS FOR CASES G–I WITH $\tau_1 = 4$, USING SIX-TERM POLYNOMIAL

μ	0	0.1	0.3	0.5	0.7	0.9	1.0	$F^{+,-}$
				Case G $\tau_1 = 4$, $\omega_0 = 1$, $\mu_0 = 1$				
R	0.623 61	0.661 81	0.698 63	0.705 77	0.697 05	0.679 31	0.668 41	0.690 92
R_0	0.593 10	0.657 37	0.698 62	0.705 98	0.697 23	0.679 46	0.668 55	0.690 93
Δ	0.030 51	0.004 44	0.000 01	−0.000 21	−0.000 18	−0.000 15	−0.000 14	−0.000 01
T	0.142 76	0.167 11	0.216 10	0.262 18	0.302 22	0.333 42	0.345 48	0.309 07
T_0	0.132 28	0.164 76	0.216 05	0.262 30	0.302 34	0.333 52	0.345 58	0.309 07
Δ	0.010 48	0.002 35	0.000 05	−0.000 12	−0.000 12	−0.000 10	−0.000 10	0.000 00
				Case H $\tau_1 = 4$, $\omega_0 = 1$, $\mu_0 = 0.5$				
R	0.732 97	0.686 99	0.606 59	0.541 56	0.488 84	0.445 34	0.426 31	0.504 14
R_0	0.698 77	0.682 18	0.606 52	0.541 72	0.488 99	0.445 45	0.426 41	0.504 09
Δ	0.043 20	0.004 81	0.000 07	−0.000 16	−0.000 15	−0.000 11	−0.000 10	0.000 05
T	0.032 44	0.035 27	0.045 28	0.058 43	0.073 66	0.088 66	0.095 47	0.071 84
T_0	0.027 12	0.033 62	0.045 18	0.058 49	0.073 72	0.088 72	0.095 52	0.071 82
Δ	0.005 32	0.001 65	0.000 10	−0.000 06	−0.000 06	−0.000 06	−0.000 05	0.000 02
				Case I $\tau_1 = 4$, $\omega_0 = 1$, $\mu_0 = 0.3$				
R	1.327 14	1.186 15	0.999 49	0.880 57	0.794 85	0.727 60	0.698 63	0.825 01
R_0	1.292 99	1.185 98	0.999 64	0.880 57	0.794 84	0.727 59	0.698 62	0.825 02
Δ	0.034 15	0.000 17	−0.000 15	0.000 00	0.000 01	0.000 01	0.000 01	−0.000 01
T	0.071 63	0.092 75	0.124 73	0.153 55	0.181 22	0.205 67	0.216 10	0.174 97
T_0	0.075 88	0.094 68	0.124 80	0.153 46	0.181 14	0.205 60	0.216 05	0.174 98
Δ	−0.004 25	−0.001 93	−0.000 07	0.000 09	0.000 08	0.000 07	0.000 05	−0.000 01

Note. Values of R_0 and T_0 are taken from van de Hulst (1980).

TABLE IV

REFLECTIVITY (R) AND TRANSMISSION (T) FOR CONSERVATIVE INFINITE ATMOSPHERE (CASE J)

μ	R	R_0	Δ
0	0.726 9193	0.726 9527	−0.000 0334
0.1	0.824 3332	0.824 3314	0.000 0018
0.2	0.878 6156	0.878 6140	0.000 0016
0.3	0.918 4890	0.918 4891	−0.000 0001
0.4	0.949 8546	0.949 8548	−0.000 0002
0.5	0.975 4632	0.975 4633	−0.000 0001
0.6	0.996 8943	0.996 8942	0.000 0001
0.7	1.015 1573	1.015 1573	0.000 0000
0.8	1.030 9418	1.030 9418	0.000 0000
0.9	1.044 7410	1.044 7410	0.000 0000
1.0	1.056 9203	1.056 9203	0.000 0000

Note. The values for R are computed by us using the parameters obtained by Huang (1952b); those for R_0 are computed using the results of Placzek (1947).

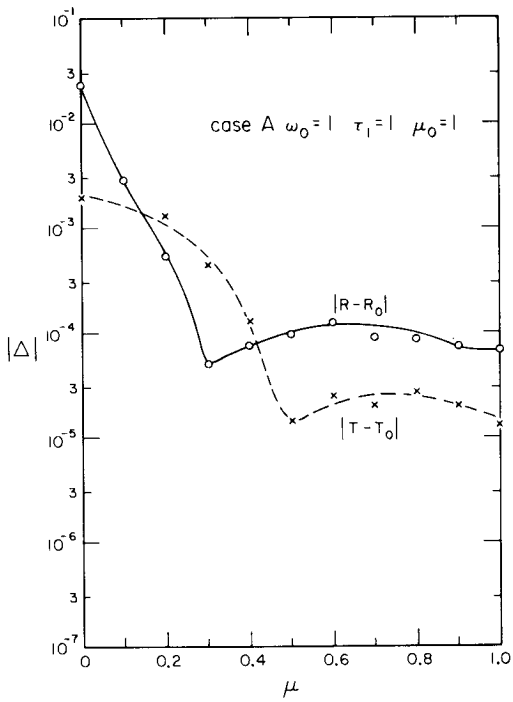

FIG. 2. Variation of $|\delta R|$ and $|\delta T|$ with μ for case A of Table I. Both $|\delta R|$ and $|\delta T|$ go through a change in sign at around $\mu = 0.3$ and therefore the dip in the graph. Numerical values of R_0 and T_0 are taken from van de Hulst (1980) for odd values of μ and from Carlstedt and Mullikin (1966) for even values of μ.

gure is probably swamped by round-off error.

Table V shows the results of an alternative use of the functional. Instead of deriving the source function directly from the functional, Sze's method (Sze, 1976) of a N-step approximation ($N = 3$ in our case) and then iteration was used. The iterated J and K were finally fitted to a fifth-order polynomial for numerical integration. Since the $J(t)$ and $K(t)$ used are not directly derived from the function, R_1, T_1 and R_2, T_2 are no longer equal to R_3, T_3 (to second order). The data demonstrated that the error of the functional method is significantly smaller than that of direct convolution.

The variation of error with μ for a polynomial trial function is shown graphically in Fig. 2 (for $\tau_1 = 1$) and Fig. 3 (for $\tau_1 = 4$). The dip in the curve at around $\mu \sim 0.3$ is not a true minimum, but a zero crossing when Δ changes sign as μ is decreased.

Figures 4 and 5 show the improvement of the functional over direct convolution. We

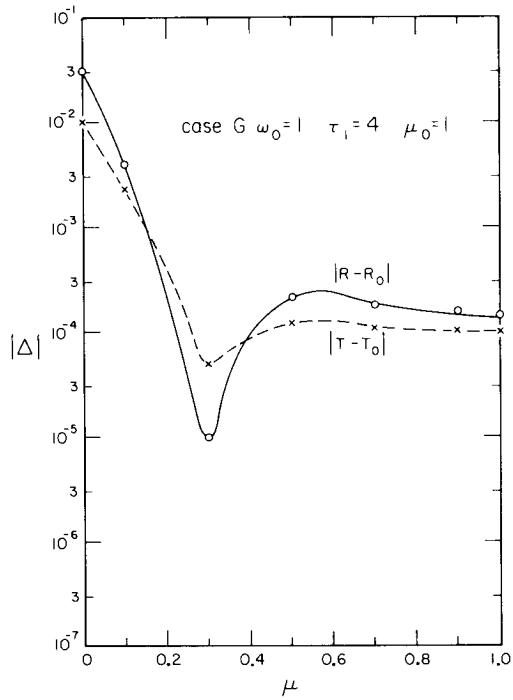

FIG. 3. Same as Fig. 2 except $\tau_1 = 4.0$.

TABLE V

THREE DIFFERENT EXPRESSIONS FOR EVALUATING THE REFLECTION AND TRANSMISSION FUNCTIONS AS DEFINED BY EQ. (10) ARE COMPARED

μ	0.1	0.3	0.5	0.7	0.9	1.0
			Case K $\tau_1 = 1$, $\omega_0 = 1$, $\mu_0 = 1$			
R_1	0.483 28	0.437 84	0.379 03	0.326 98	0.285 12	0.267 54
Δ_1	0.018 40	0.001 18	−0.001 48	−0.001 93	−0.001 91	−0.001 85
R_2	0.462 95	0.433 58	0.377 79	0.326 59	0.285 04	0.267 54
Δ_2	−0.001 93	−0.003 08	−0.002 72	−0.002 32	−0.001 99	−0.001 85
R_3	0.465 21	0.436 66	0.380 48	0.328 87	0.286 99	0.269 35
Δ_3	0.000 33	0.000 00	−0.000 03	−0.000 04	−0.000 04	−0.000 04
T_1	0.316 88	0.340 53	0.321 20	0.289 49	0.259 04	0.245 32
Δ_1	0.023 48	0.004 17	0.000 29	−0.000 81	−0.001 14	−0.001 21
T_2	0.296 34	0.336 11	0.319 86	0.289 07	0.258 95	0.245 32
Δ_2	0.002 94	−0.000 25	−0.001 05	−0.001 23	0.001 23	−0.001 21
T_3	0.293 90	0.336 45	0.320 93	0.290 29	0.260 17	0.246 51
Δ_3	0.000 50	0.000 09	0.000 02	−0.000 01	−0.000 01	−0.000 02
			Case L $\tau_1 = 1$, $\omega_0 = 1$, $\mu_0 = 0.1$			
R_1	1.728 16	1.042 77	0.764 31	0.605 88	0.502 40	0.462 95
Δ_1	−0.051 48	−0.015 67	−0.007 51	−0.004 14	−0.002 46	−0.001 93
R_2	1.728 16	1.067 39	0.790 53	0.629 91	0.523 90	0.483 28
Δ_2	−0.051 48	0.008 95	0.018 71	0.019 89	0.019 04	0.018 40
R_3	1.769 66	1.056 98	0.771 66	0.610 23	0.505 18	0.465 21
Δ_3	−0.009 98	−0.001 46	−0.000 16	0.000 21	0.000 32	0.000 33
T_1	0.209 81	0.286 76	0.326 19	0.323 90	0.306 68	0.296 34
Δ_1	0.032 27	0.013 05	0.007 56	0.004 92	0.003 44	0.002 94
T_2	0.209 81	0.309 72	0.351 48	0.347 65	0.328 28	0.316 88
Δ_2	0.032 27	0.036 01	0.032 85	0.028 67	0.025 04	0.023 48
T_3	0.175 50	0.273 85	0.319 13	0.319 52	0.303 75	0.293 89
Δ_3	−0.002 04	0.000 14	0.000 50	0.000 54	0.000 51	0.000 49

Note. The trial functions are obtained in three steps. First the variation principle using three-step functions is used. The result is refined by one iteration (Sze, 1976) using the integral equation, and finally fitted to a fifth-order polynomial.

started with a three-step approximation and the numerical integration was done using a fifth-order polynomial as described earlier. Δ stands for the difference as in Table II, its improvement ranges from a factor of 2 to two orders of magnitude.

Another case of interest is that of a Lambert surface (with albedo λ) at the lower boundary of the atmosphere. In this case, the functions $S(s,\mu)$ and $G(t,s)$ in Eq. (5) are modified to

$$S(s,\mu) = \sqrt{\omega_0(s)} \left[\frac{F}{4\mu} e^{-s/\mu} + \frac{\lambda F}{2} e^{-\tau_1/\mu_0} E_2(\tau_1 - s)\right]$$

$$G(t,s) = \frac{\sqrt{\omega_0(t)\omega_0(s)}}{2} [E_1|t - s| + 2\lambda E_2(\tau_1 - t)E_2(\tau_1 - s)].$$

Otherwise, the entire calculation goes through as before. We obtained results with similar degree of accuracy as in the case $\lambda = 0$.

CONCLUSION

We have provided a generalization of the method of Stokes and DeMarcus (1971) to calculate reflectivity and transmissivity at arbitrary emergent angles. For the special

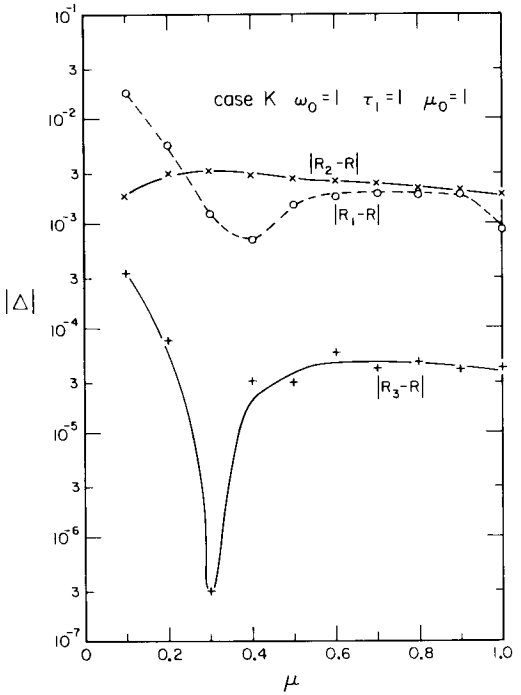

FIG. 4. Comparison of accuracy of the three methods of evaluating the reflectivity using the three-step approximate J as in Table V. Note that the functional method can be more accurate by one to two orders of magnitude.

case of trial functions linear in the variational parameters there is a degeneracy between the Stokes and DeMarcus method and ours. Error analysis has demonstrated the difference between the two independent uses of the functional. If one uses trial functions of the form given by (12) and derives the mean intensity J from the functional, then the reflectivity and transmissivity can be computed to second order by direct convolution. It is unnecessary to evaluate K or the full functional. If one uses a J (and K) not directly derived from the functional then the functional gives better results than direct convolution. There is a natural limitation on the accuracy of such methods in the sense that errors become significant for $\mu \lesssim 0.2$. This is to be expected since small μ is equivalent to large effective optical depth and the simple trial functions used in the sample calculations are insufficient.

The construction of our functional was motivated by Levine and Schwinger's (1950) variational principles of electromagnetic wave diffraction. The theory of radiative transfer is closely related to the theory of neutron diffusion. Therefore it is not surprising to detect striking similarity of our functional with the one given by Davison (1957). The relation between the functional and simple physical quantities, first discovered by Stokes and DeMarcus (1971), has been extended by us. Although this paper does not significantly advance the computational aspects of the planetary problem beyond Sze (1976), it has brought new insight—the connection between the principle of reciprocity and the variational method—to the problem of radiative transfer. Independent work by Cheyney and Arking (1976) and Shia and Yung (personal communication, 1984) indicates that generalization to problems with nontrivial phase function and non-plane-parallel geometry can be made.

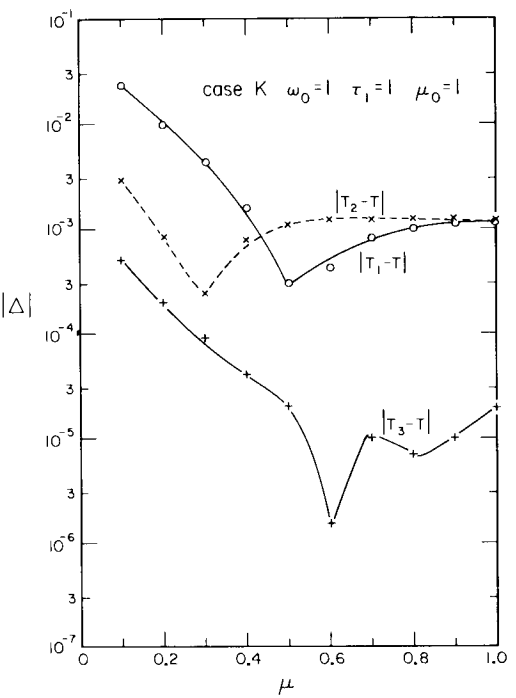

FIG. 5. Same as Fig. 4, for transmissivity.

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